

Perspectives for relativistic many-body and QED calculations for highly charged ions

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- Introduction: many body calculations in atoms and ions
- Exemple of applications to new high-precision experiments
 - few-electron uranium ions
 - medium- Z few electron ions
 - long-lived metastable states
- Conclusion and perspective

Many body effects

principles and examples

$$\mathcal{H}^{\text{no pair}} \Psi_{\Pi, J, M}(\dots, \mathbf{r}_i, \dots) = E_{\Pi, J, M} \Psi_{\Pi, J, M}(\dots, \mathbf{r}_i, \dots),$$

$$\mathcal{H}^{\text{no pair}} = \sum_{i=1}^N \mathcal{H}_D(\mathbf{r}_i) + \sum_{i < j} \mathcal{V}(|\mathbf{r}_i - \mathbf{r}_j|), \quad \mathcal{V}(|\mathbf{r}_i - \mathbf{r}_j|) = \Lambda^{++} V_{ij} \Lambda^{++},$$

$$|\Psi_{\Pi, J, M}\rangle = \sum_{v=1}^n c_v |v, \Pi, J, M\rangle.$$

c_v obtained by diagonalisation of Hamiltonian matrix

$$|v, \Pi, J, M\rangle = \sum_{i=1}^{N_v} d_i \begin{vmatrix} \psi_1^i(r_1) & \cdots & \psi_m^i(r_1) \\ \vdots & \ddots & \vdots \\ \psi_1^i(r_m) & \cdots & \psi_m^i(r_m) \end{vmatrix}.$$

d_i obtained by diagonalisation of J^2 and J_z

$$E_{\Pi,J,M} = \frac{\langle \Psi_{\Pi,J,M} | \mathcal{H}^{\text{no pair}} | \Psi_{\Pi,J,M} \rangle}{\langle \Psi_{\Pi,J,M} | \Psi_{\Pi,J,M} \rangle}$$

$$\begin{pmatrix} V_{\text{DF}}^i(r) & -\frac{d}{dr} + \frac{\kappa_i}{r} \\ \frac{d}{dr} + \frac{\kappa_i}{r} & V_{\text{DF}}^i(r) - 2mc \end{pmatrix} \begin{pmatrix} P_i(r) \\ Q_i(r) \end{pmatrix}$$

$$= \alpha \lambda_{i,i} \begin{pmatrix} P_i(r) \\ Q_i(r) \end{pmatrix} + \begin{pmatrix} X_P^i(r) \\ X_Q^i(r) \end{pmatrix} + \sum_{j \neq i} \lambda_{j,i} \begin{pmatrix} P_j(r) \\ Q_j(r) \end{pmatrix}$$

Minimize



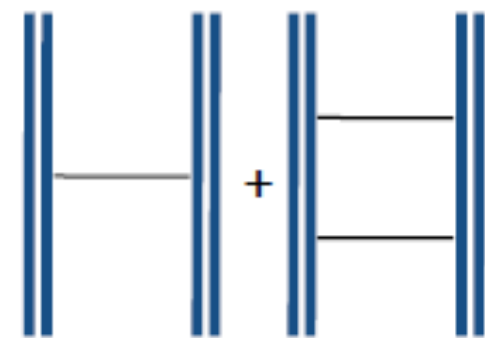
Inhomogeneous Dirac equation

- Build the energy expression with all one-electron and F, G, R integrals
- add the magnetic and retardation integrals
- Diagonalize the hamiltonian matrix
- solve each differential equation in turn up to all have reached some predefined accuracy (10^{-4} to 10^{-6})
- Diagonalize again and start an iterative solving cycle several times
- Check the energy numerical accuracy by evaluating it with two different expressions that **give the same value only if wavefunction and energy are well converged**

Electron-electron interaction

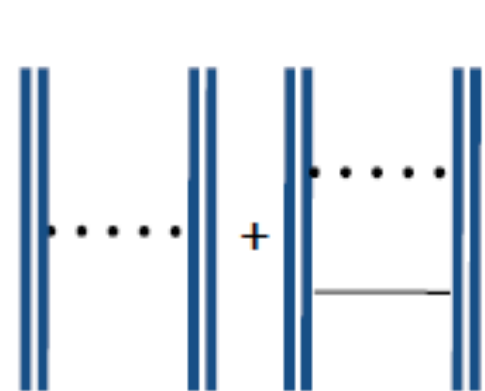


$$: \frac{1}{R} - \frac{\alpha_1 \cdot \alpha_2}{R} \cos \omega R + (\alpha_1 \cdot \partial_1)(\alpha_2 \cdot \partial_2) \left(\frac{\cos \omega R - 1}{\omega^2 R} \right)$$



$$\frac{1}{R}$$

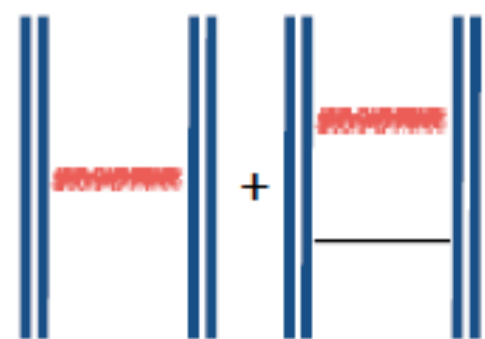
Coulomb: definition of the wave function



$$- \frac{\alpha_1 \cdot \alpha_2}{R}$$

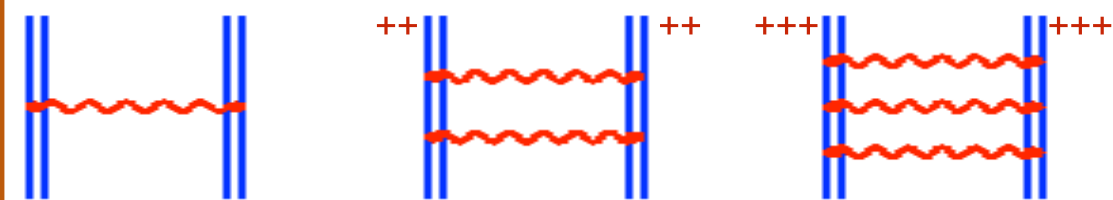
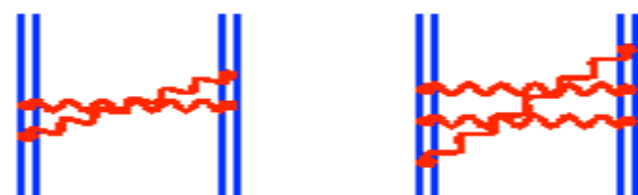
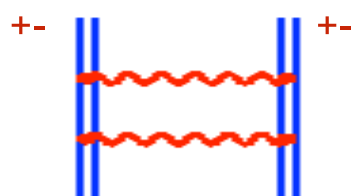
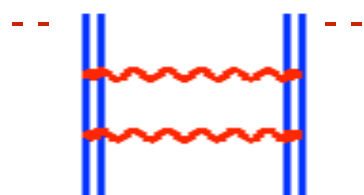
Magnetic interaction ($\omega=0$): perturbation or average potential in differential equation

a complex operator involving exponential functions of the one electron Hamiltonian.

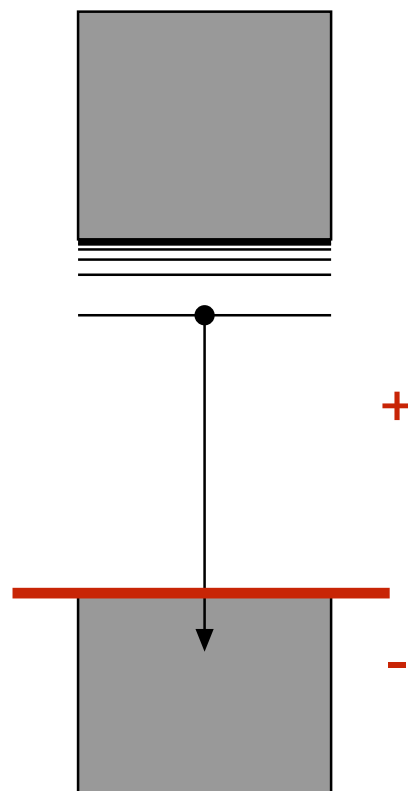


$$- \frac{\alpha_1 \cdot \alpha_2}{R} (\cos \omega R - 1) + (\alpha_1 \cdot \partial_1)(\alpha_2 \cdot \partial_2) \left(\frac{\cos \omega R - 1}{\omega^2 R} \right)$$

Retardation: the rest with $\omega = E_a - E_b$

RMBPT or MCDF or RCI ($1/Z$ expansion!)not included
in any calculationnot included
in any calculationnot included
in any calculation

+ means remove below



Challenges:

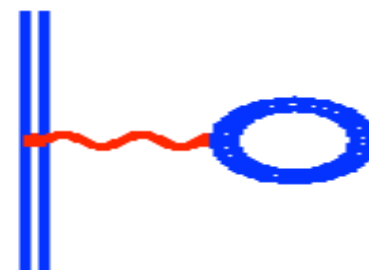
- numerical convergence of very non-linear differential equation system
- evaluation of higher-order effects (three-body for example)
 - include double and triple excitations
 - include all single excitation
 - How to take into account high-order retardation? (value of ω , numerical instabilities)
- Brillouin single excitations to get correct fine structure at low- Z ...
- Huge number of Coulomb and Breit integral for cases with many open shells with d, f, g orbitals

One electron BSQED corrections

QED, nuclear size corrections...

Self Energy

$$\frac{\alpha}{\pi}$$



Vacuum Polarization

H-like “One Photon” order (α/π)Many calculations, very accurate down to $Z=1$

$$\Delta E_{SE} = \frac{\alpha}{\pi} \frac{(Z\alpha)^4}{n^3} F(Z\alpha).$$

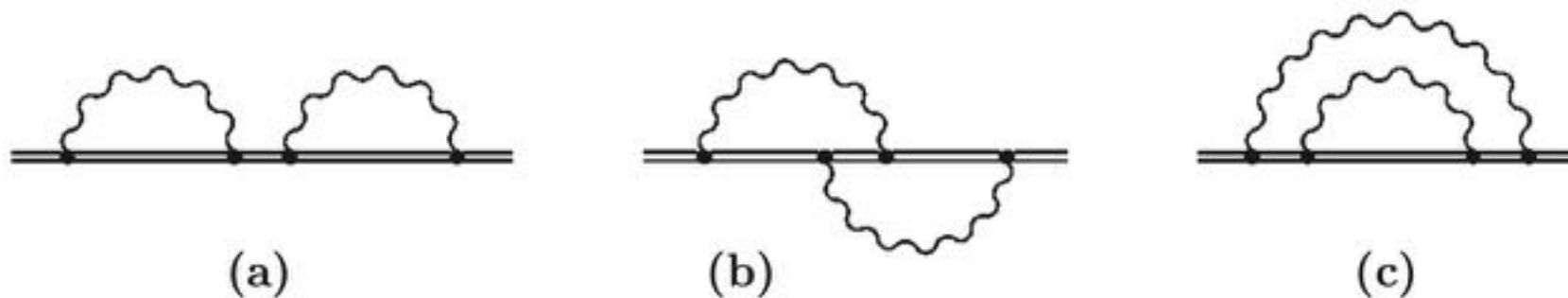
$$\frac{\alpha}{\pi} = 2.3 \times 10^{-3}$$

$$E_{NR} = \frac{(Z\alpha)^2}{n^2}$$

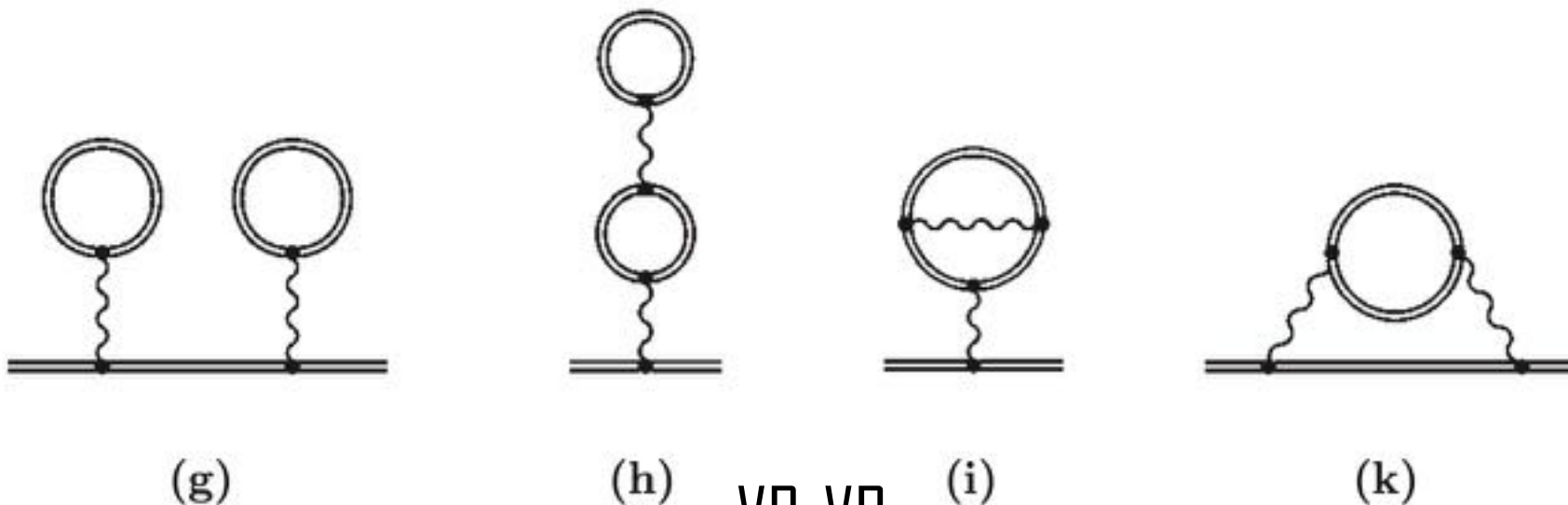
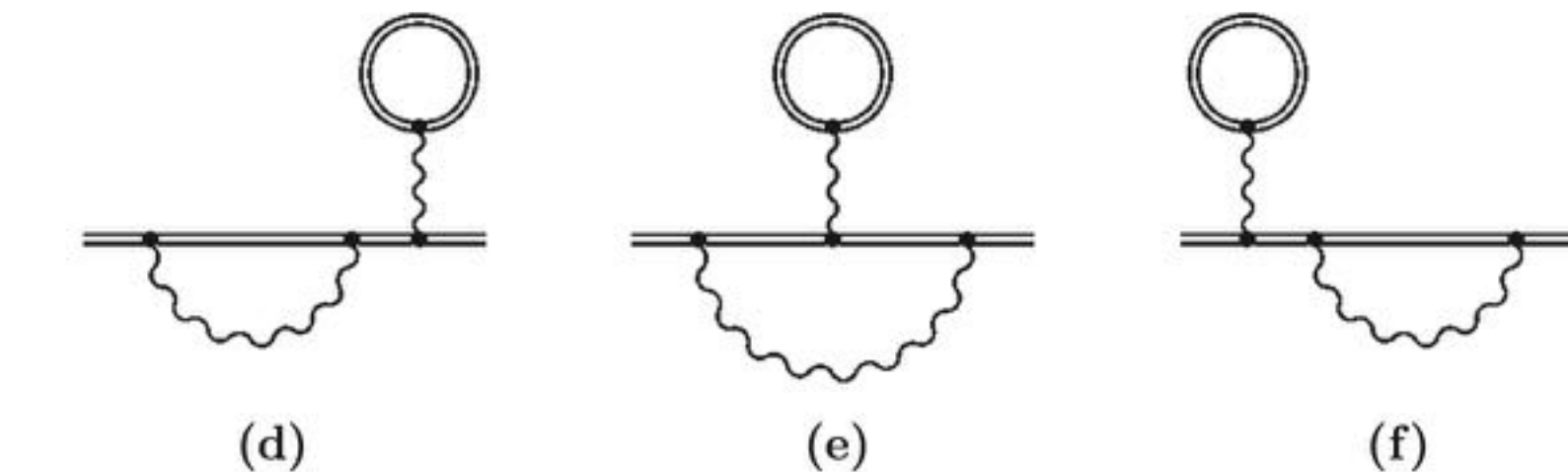
Closed fermion loops 2nd order diagrams

terms of order $(\alpha/\pi)^2$

SE-SE



SE-VP

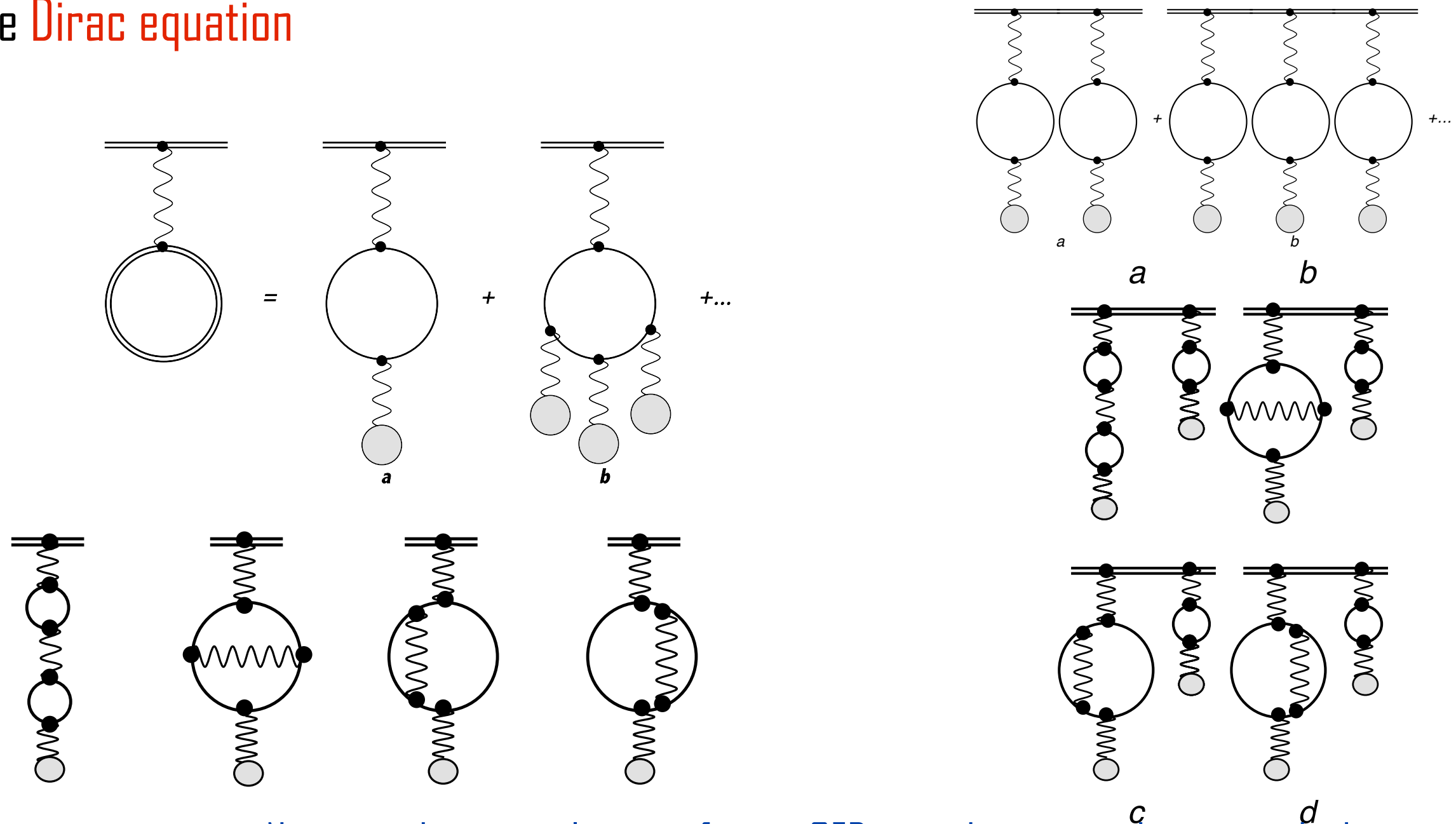


Uehling
in Dirac Eq.

vp-vp
Källén & Sabri potential

S(VP)E

All-order: the charge distribution is included exactly in the wavefunction and in the operator, when relevant. Higher order Vacuum Polarization contribution included by numerical solution of the **Dirac equation**



Nonperturbative evaluation of some QED contributions to the muonic hydrogen $n=2$ Lamb shift and hyperfine structure, P. Indelicato. Phys. Rev. A **87**, 022501 (2013).

Two-electron BSQED corrections

QED, nuclear size corrections...

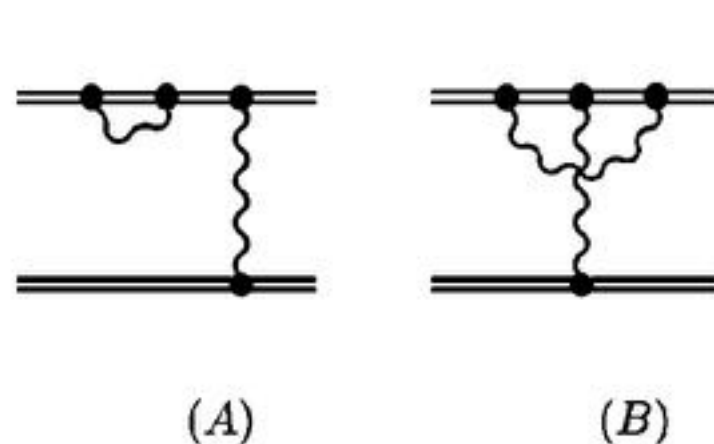
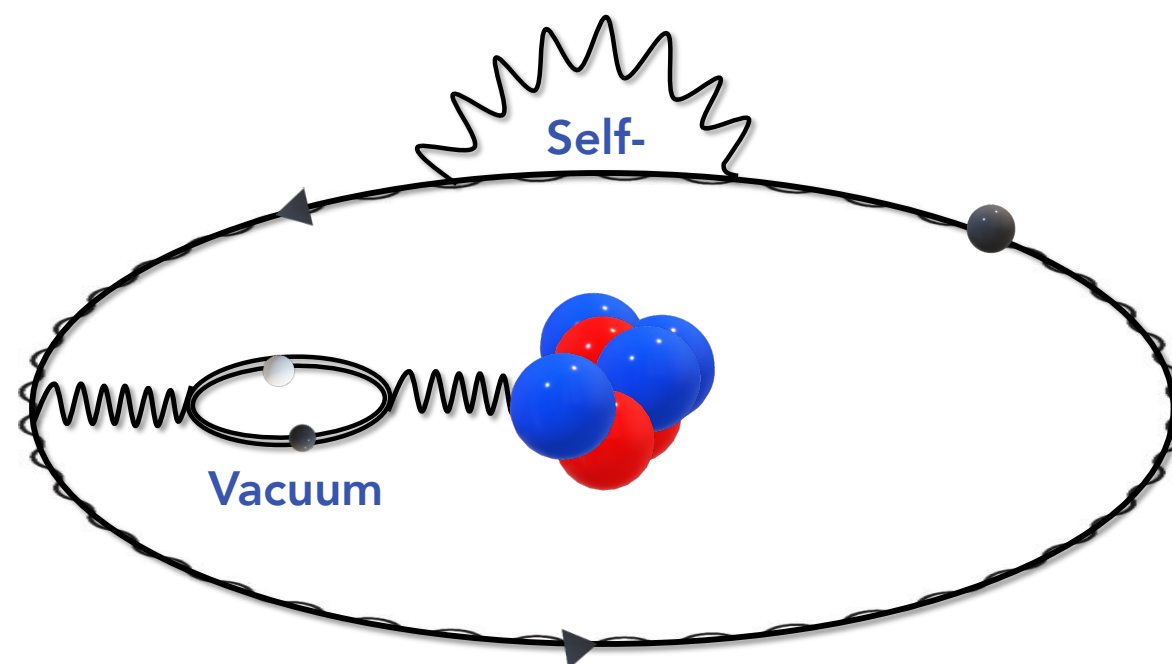


FIG. 1. Feynman diagrams for radiative corrections to the electron-electron interaction. (A) represents the wave function correction, and (B) is the vertex correction.



Direct QED calculation, e.g.:

Coordinate-space approach to the bound-electron self-energy: Self-Energy screening calculation, P. Indelicato et P.J. Mohr. Phys. Rev. A 63, 052507 (2001).

Approximation: Welton model

MCDF studies of two electron ions II: Radiative corrections and comparison with experiment., P. Indelicato, O. Gorceix et J.P. Desclaux. J. Phys. B 20, 651 (1987).

Until recently:

- Welton model $\delta H_{\text{SE}} = \Delta V_n(r) \langle (\delta r)^2 \rangle_{\text{vacuum}}$

$$\delta E_{\text{SE}} = \sum \frac{\langle nl | \Delta U_n(r) | nl \rangle_{\text{DF}}}{\langle nl | \Delta U_n(r) | nl \rangle_{\text{Hyd}}} \delta(E_{\text{SE Hyd}})_{nl}.$$

(P. Indelicato, O. Gorceix, and J. P. Desclaux, J. Phys. B: Atomic 20, 651 (1987).)

- Model potential: calculate the self-energy in a non-coulomb, screened potential

A model operator has been defined:
This operator can be easily incorporated in any calculations employing the Dirac-Coulomb-Breit Hamiltonian.

$$h^{\text{SE}} = V_{\text{loc}}^{\text{SE}} + \sum_{i,k=1}^n |\phi_i\rangle B_{ik} \langle \phi_k| ,$$

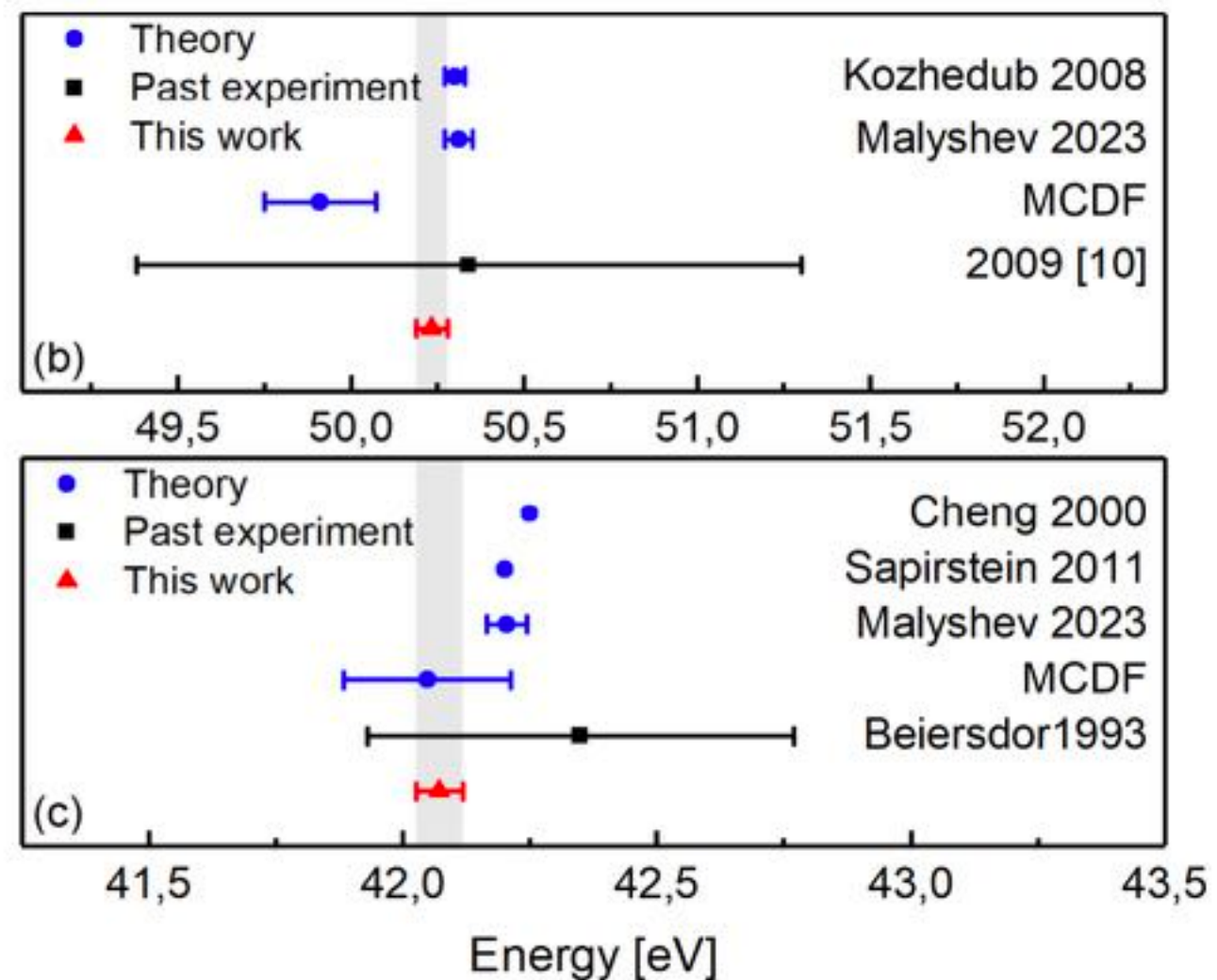
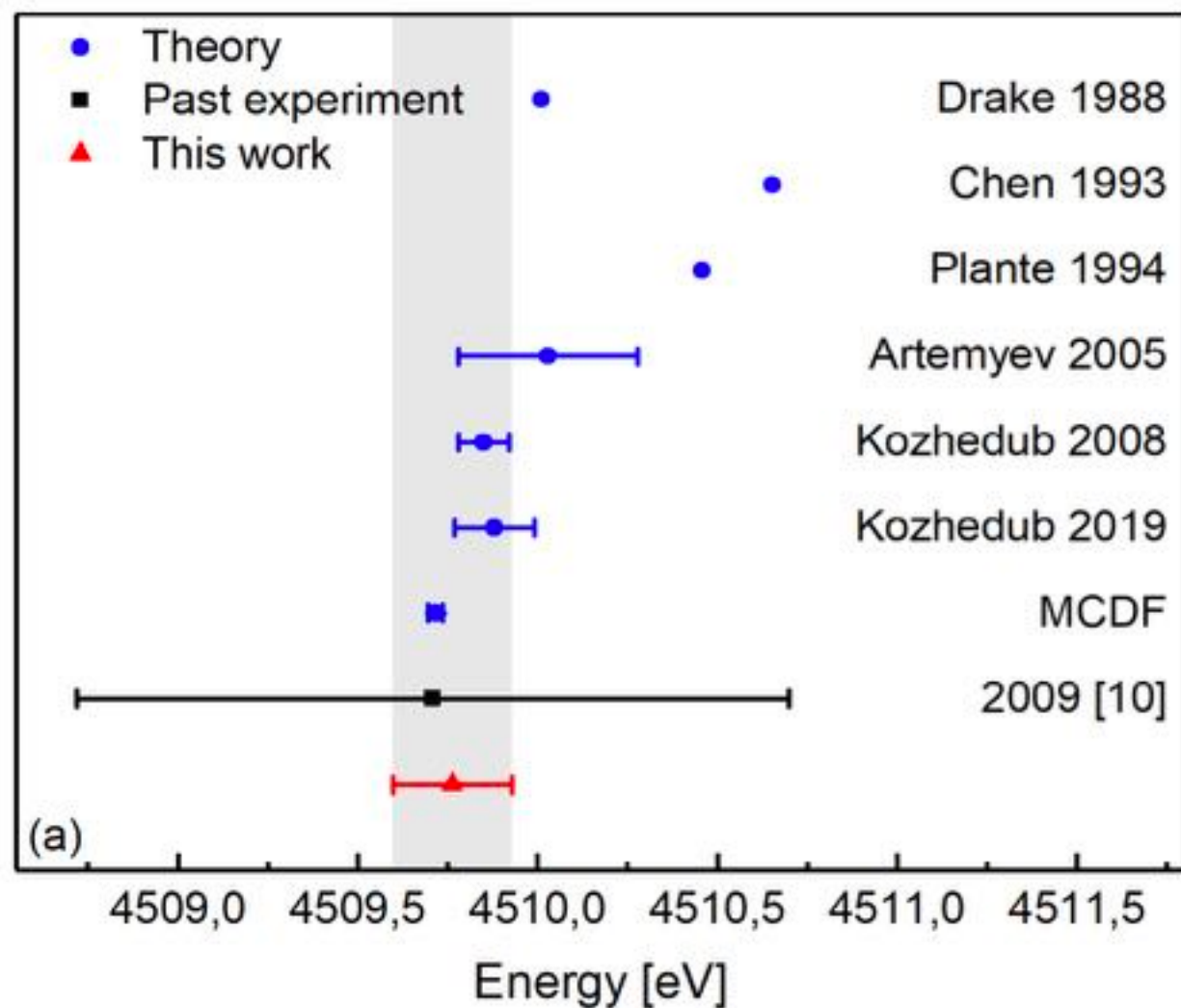
The model operator reproduces very precisely exact evaluation of the Self-energy values for one-electron ions

Now implemented since the 2016 version of the MCDFGME code (Indelicato and Desclaux)

Shabaev, V. M., I. I. Tupitsyn and V. A. Yerokhin (2013), Physical Review A88(1): 012513.

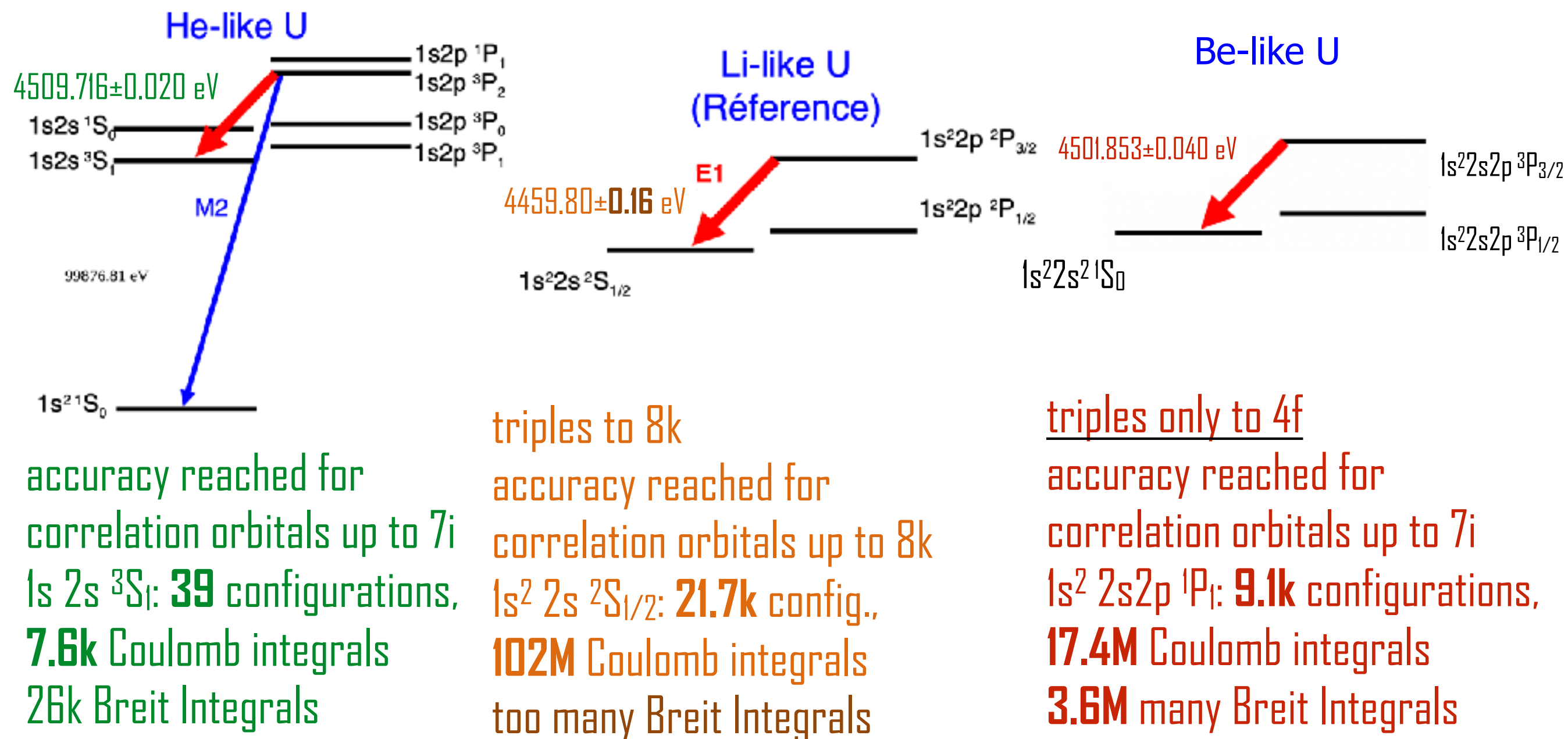
High-precision measurements $\Delta n=0$ few-electrons U ions

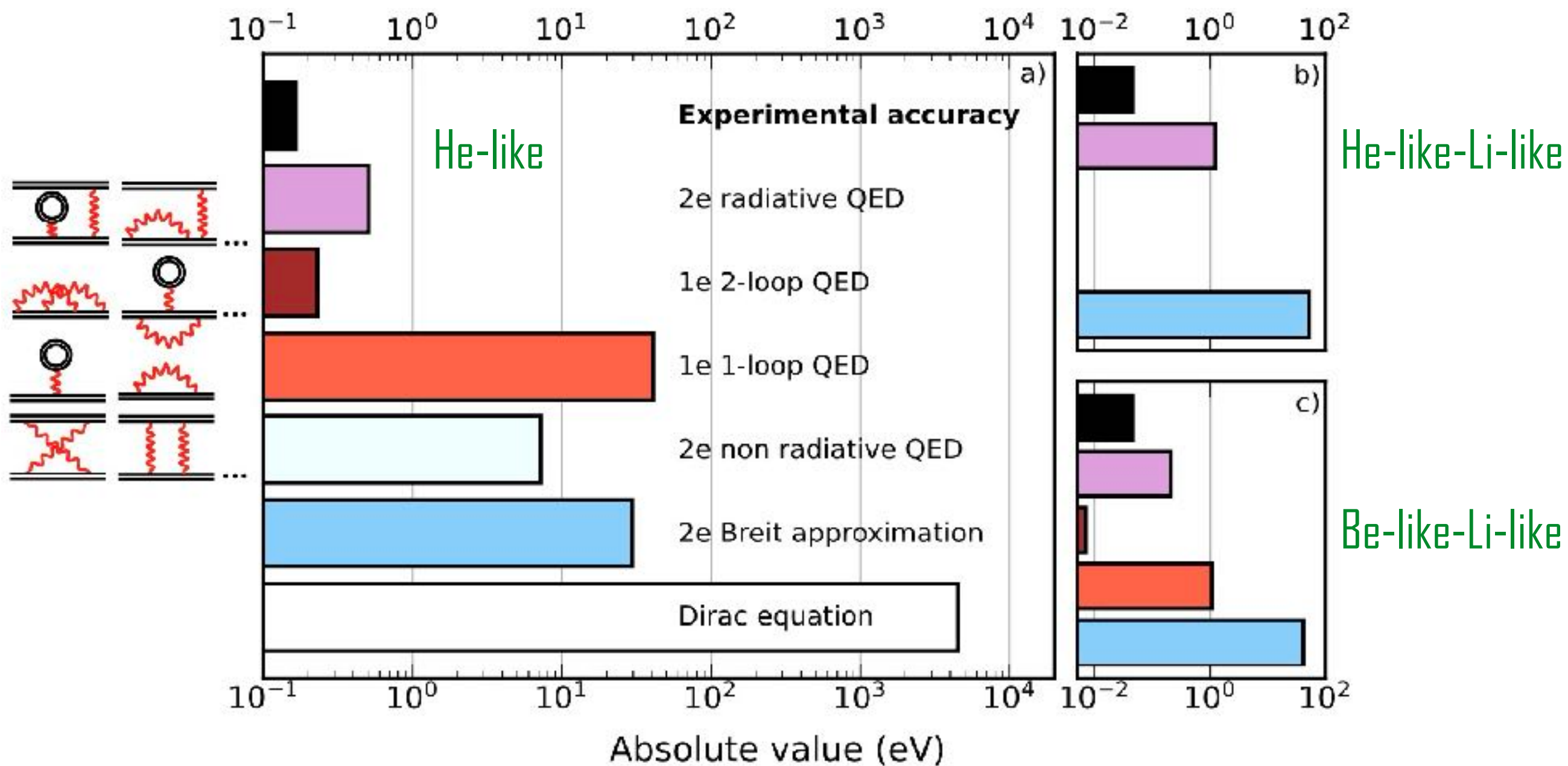
Crystal spectroscopy at ESR



R. Loetzsch, H.F. Beyer, L. Duval, U. Spillmann, D. Banas, P. Dergham, F.M. Kröger, J. Glorius, R.E. Grisenti, M. Guerra, A. Gumberidze, R. Heß, P.-M. Hillenbrand, P. Indelicato, P. Jagodzinski, E. Lamour, B. Lorentz, S. Litvinov, Yu.A. Litvinov, J. Machado, N. Paul, G.G. Paulus, G. N. Petridis, J.P. Santos, M. Scheidel, I. Uschmann, G. Weber, Th. Stöhlker, M. Trassinelli, accepted to Nature (2024)

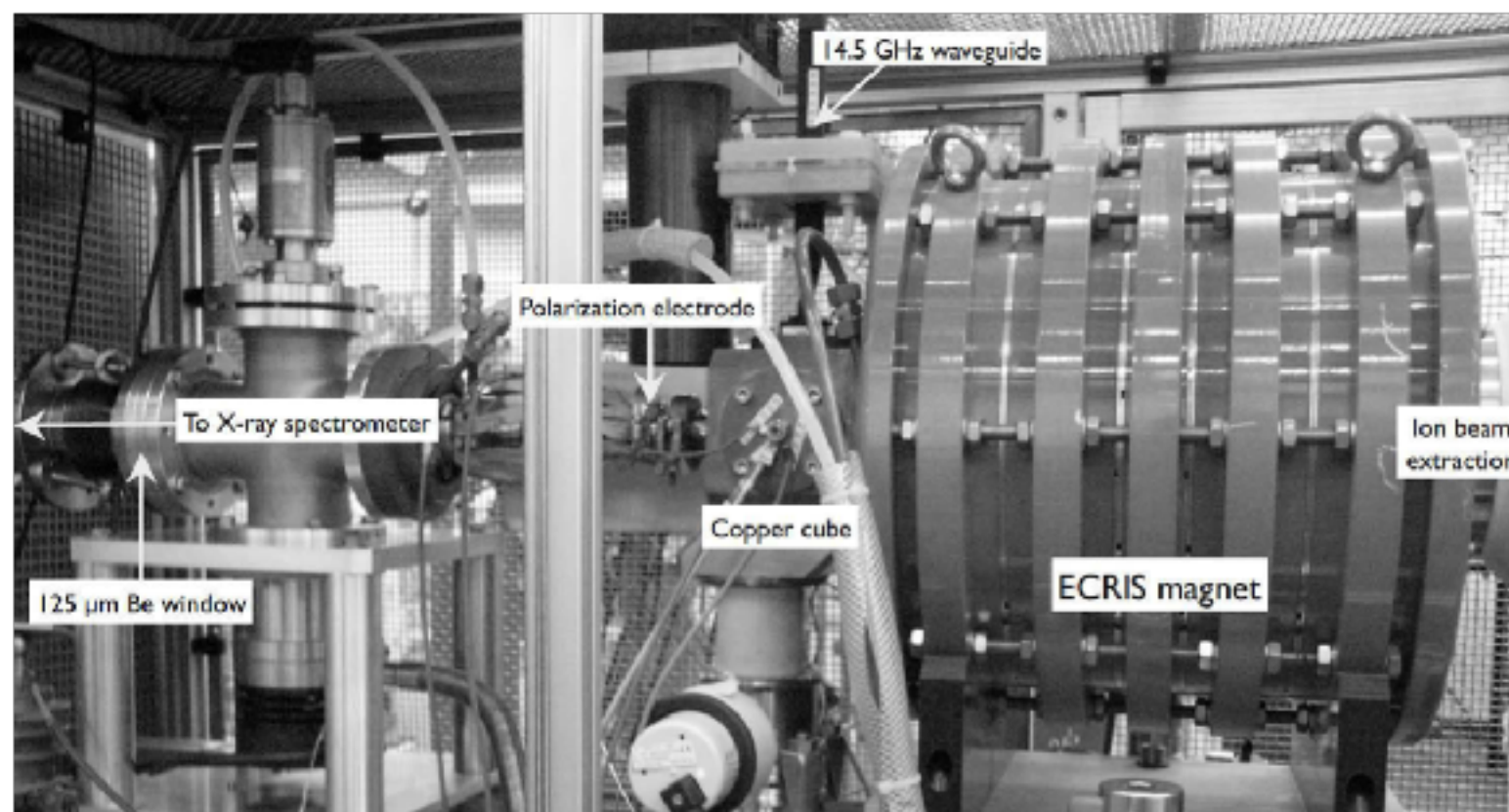
Comparing two states of charges allows to bypass the nuclear uncertainties





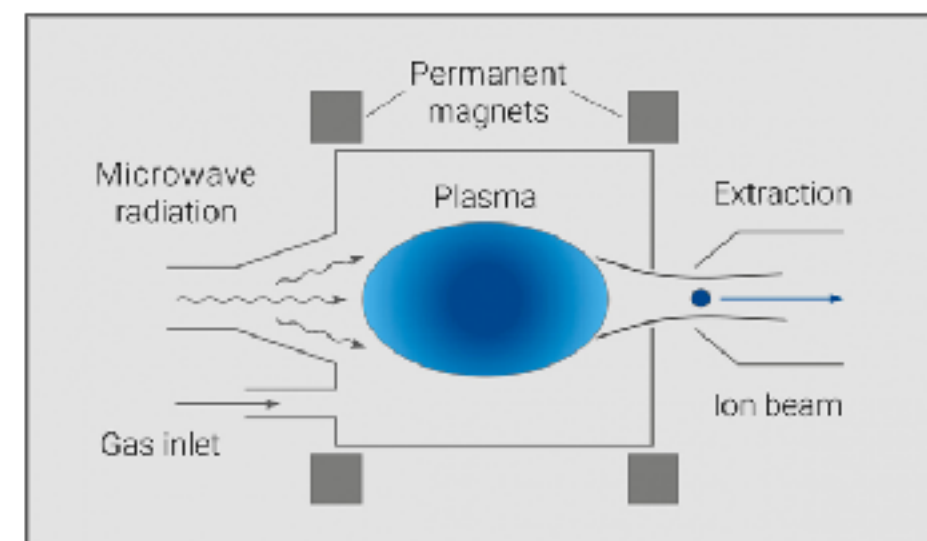
Medium-Z few-electron ions

Reference-free measurements in 2-5 ions
comparison with theory and astrophysics benchmarks



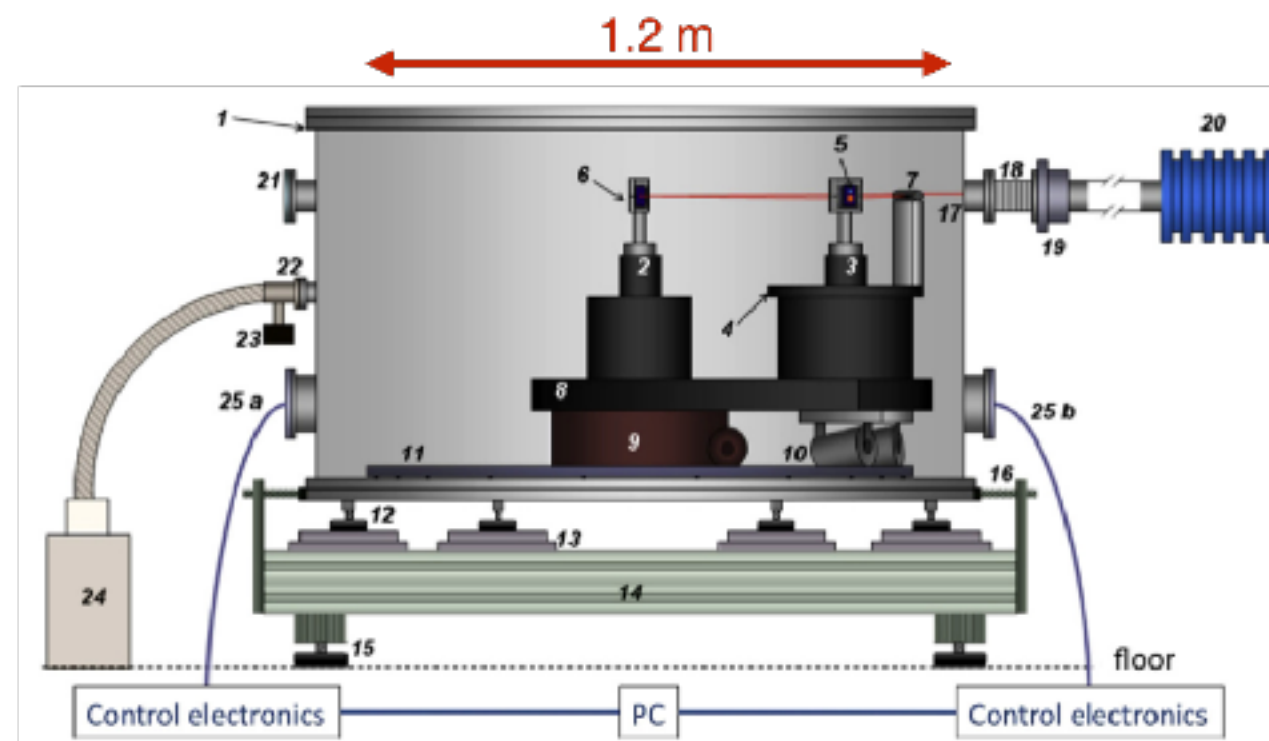
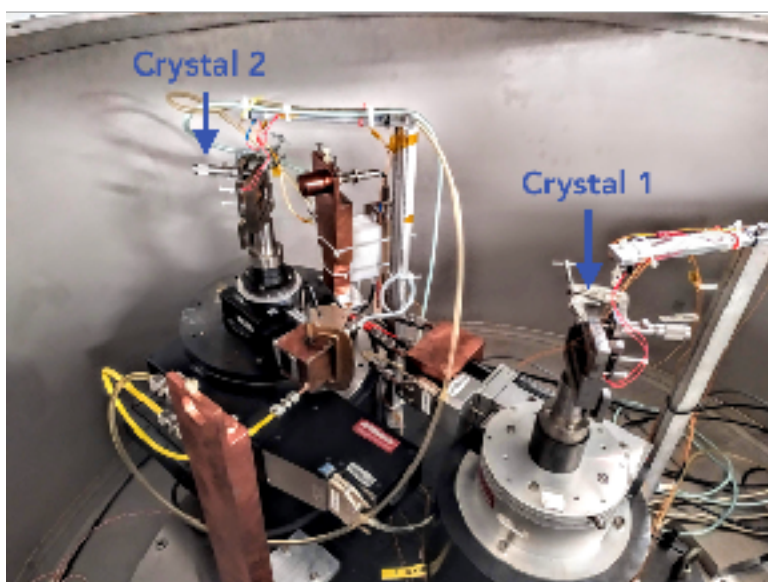
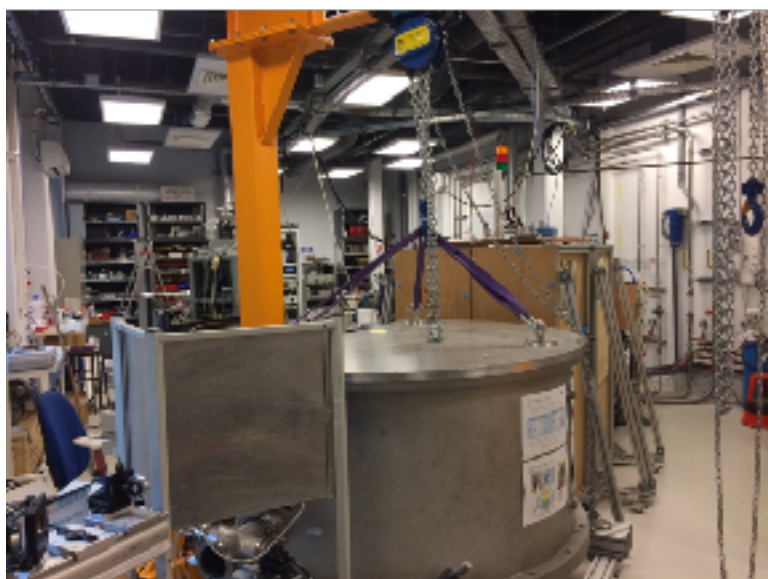
Microwaves : 14.5 GHz

Extraction voltage: 0V to 25 kV



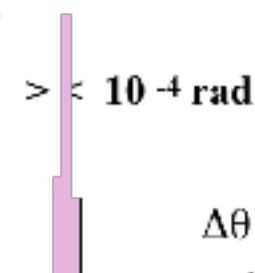
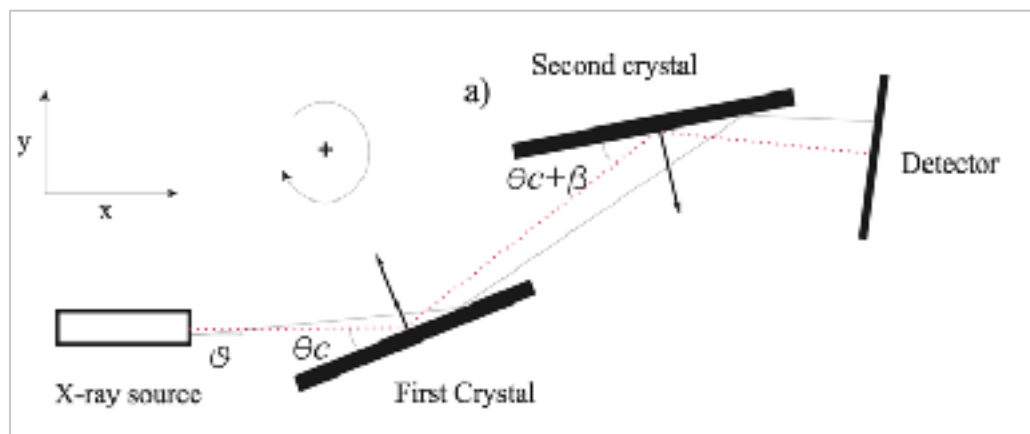
- Direct connection to plasma, 50 μm thick Be window
- In the plasma the ions are trapped in the space charge of the electrons ($\sim 10^{11}$ e/cm³), $\sim$ few eV trapping depth
- Intense source, provides access to forbidden transitions, narrow linewidths

The Paris Double Crystal Spectrometer



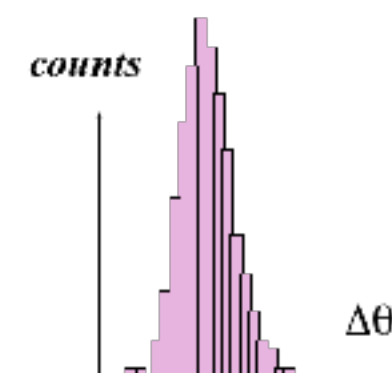
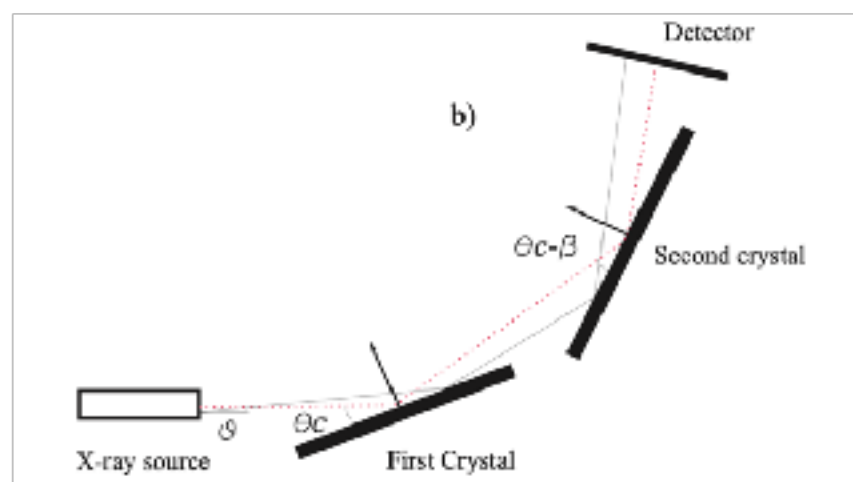
- Si₁₁₁ crystals from NIST, lattice spacing (d) known to 10^{-8}
- Angular encoder for second axis: Heidenhain RON 905 with AWE 1024 interpolator $\rightarrow 0.2''$ of arc angular accuracy
- Detector : LAAPD (large area avalanche photodiode) cooled at -10°C

Parallel
Energy
Non-dispersive

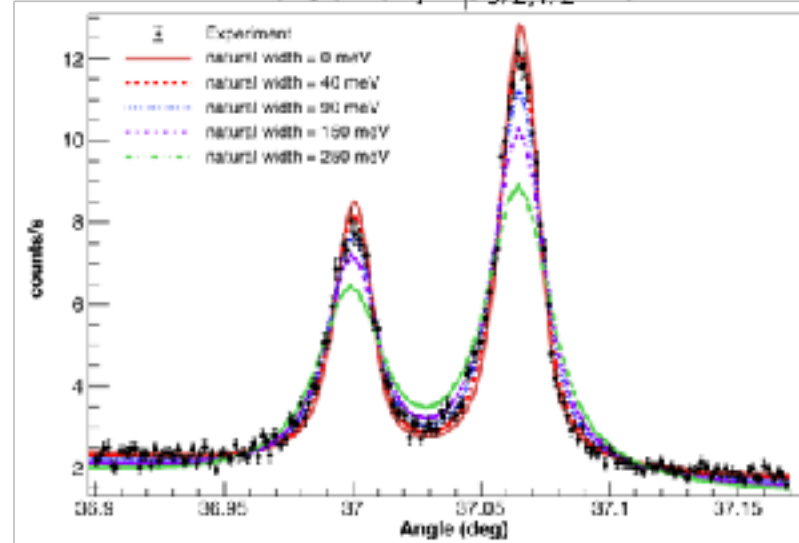
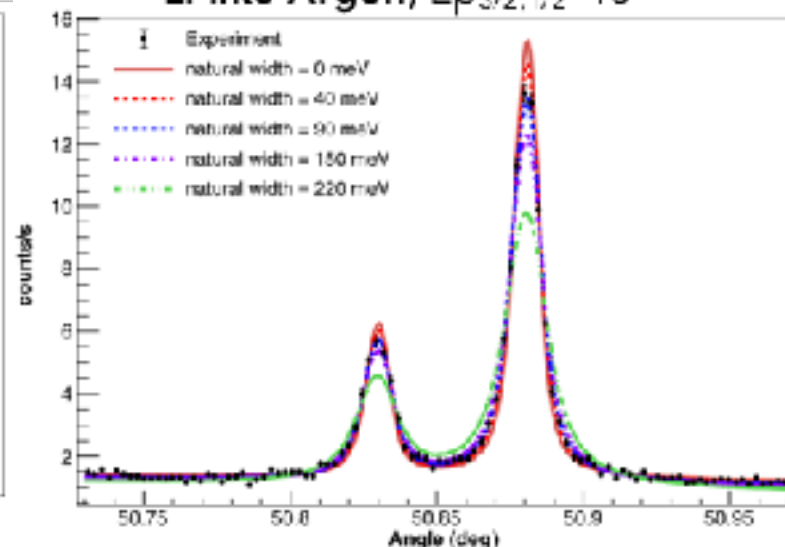


width :
DCS response function

Antiparallel
Energy Dispersive



width :
intrinsic line width
Doppler broadening
DCS response function

Li-like Sulfur, $2p_{3/2,1/2}-1s$ Li-like Argon, $2p_{3/2,1/2}-1s$ 

Highest precision, reference-free measurements in core-excited Li-like ions

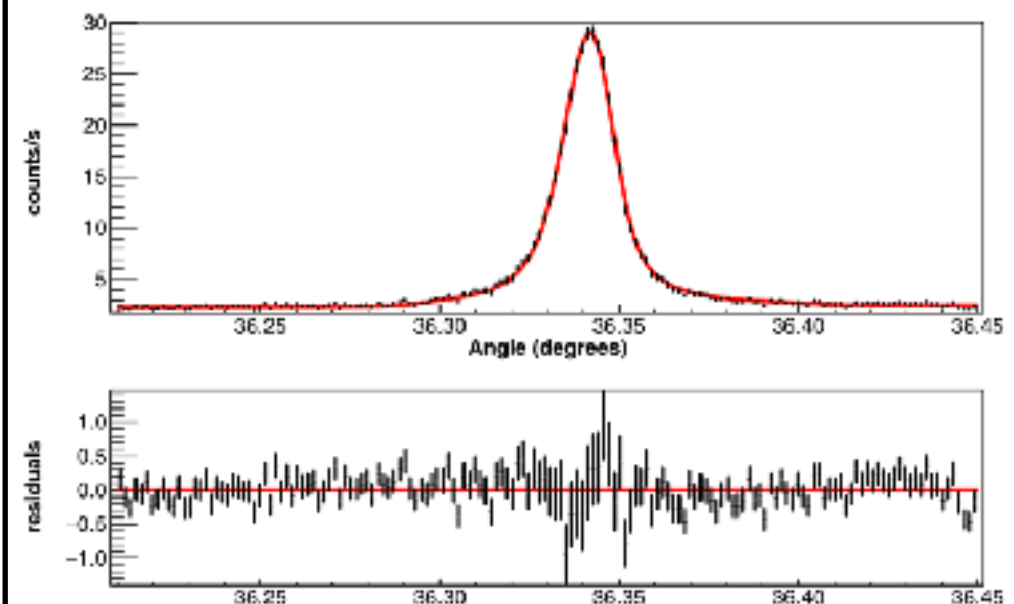
Sulfur peak ratio : 0.46 [theory], 0.627(22) [exp]

Argon peak ratio : 0.44 [theory], 0.397(14) [exp]

Cannot be explained by known contaminant lines

J. Machado, G. Bian, N. Paul, et al,
PRA 101, 062505 (2020)

He-like Sulfur



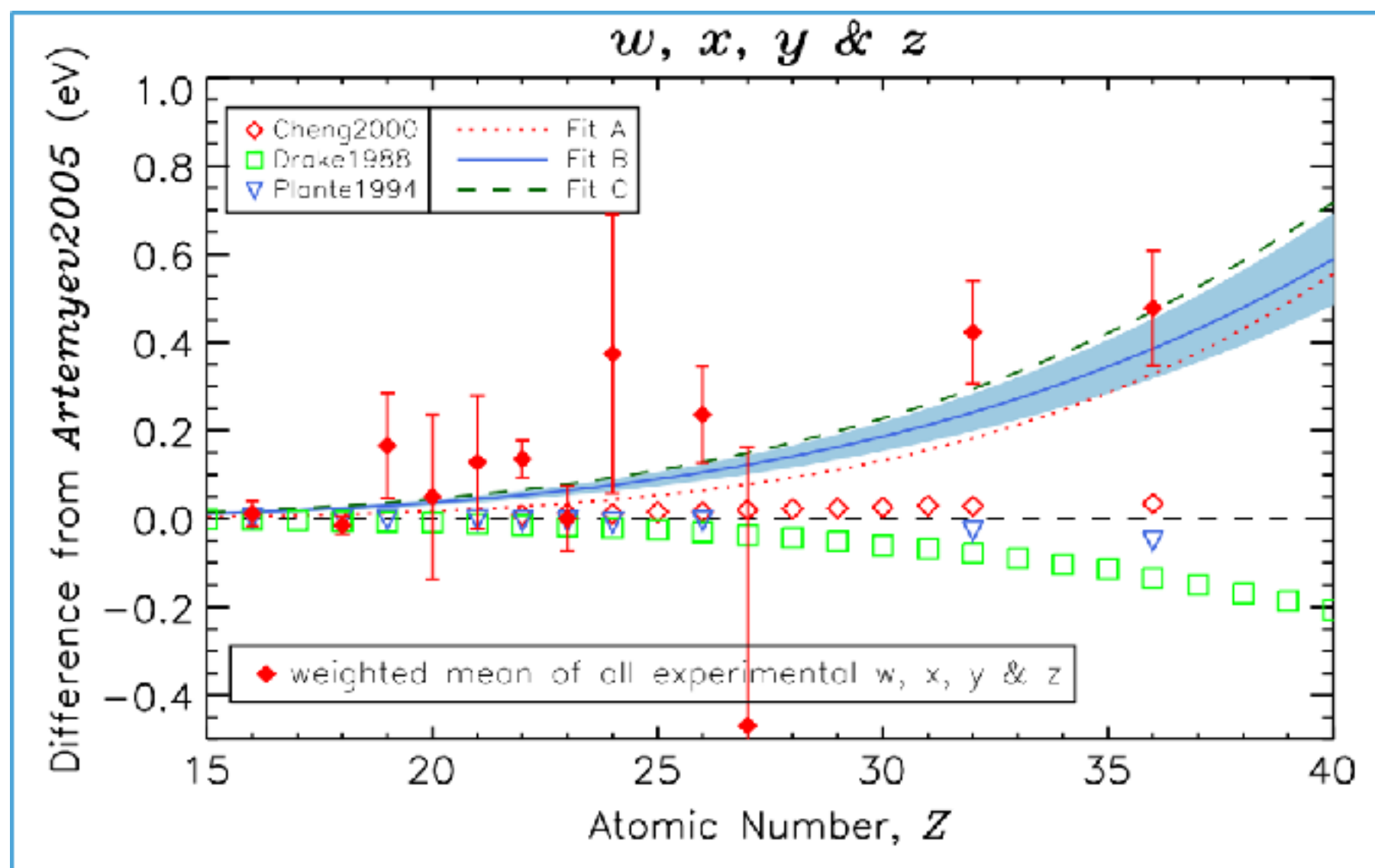
First ppm measurements of the relativistic magnetic dipole transition in He-like S

Test of 2e QED effects, sensitive to crystal form factors

J. Machado, N. Paul, et al,
PRA 107, 032821 (2023)

$$n=2 \rightarrow n=1$$

Z^4 dependence claimed for fit B

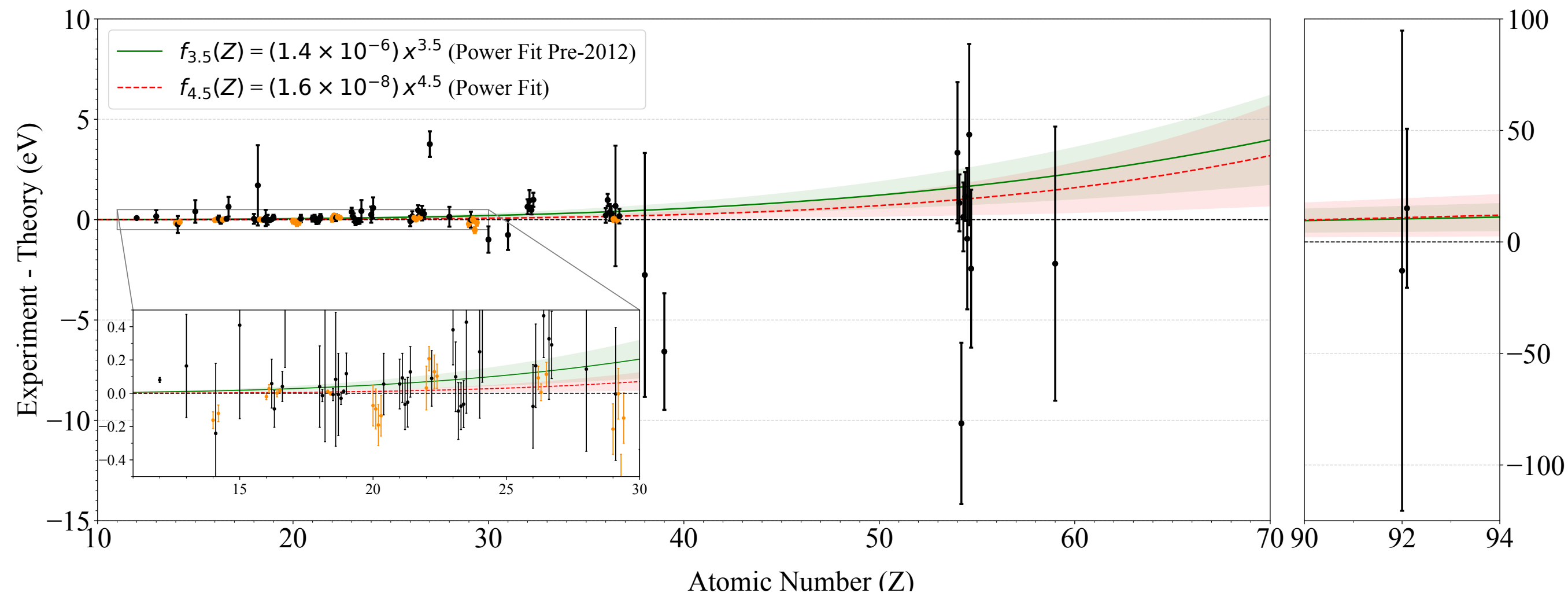


deviation at
 $Z=92 \sim 16.8$ eV

C T Chantler et al. 2014 New J. Phys. 16 123037

C T Chantler et al. 2014 New J. Phys. 16 123037

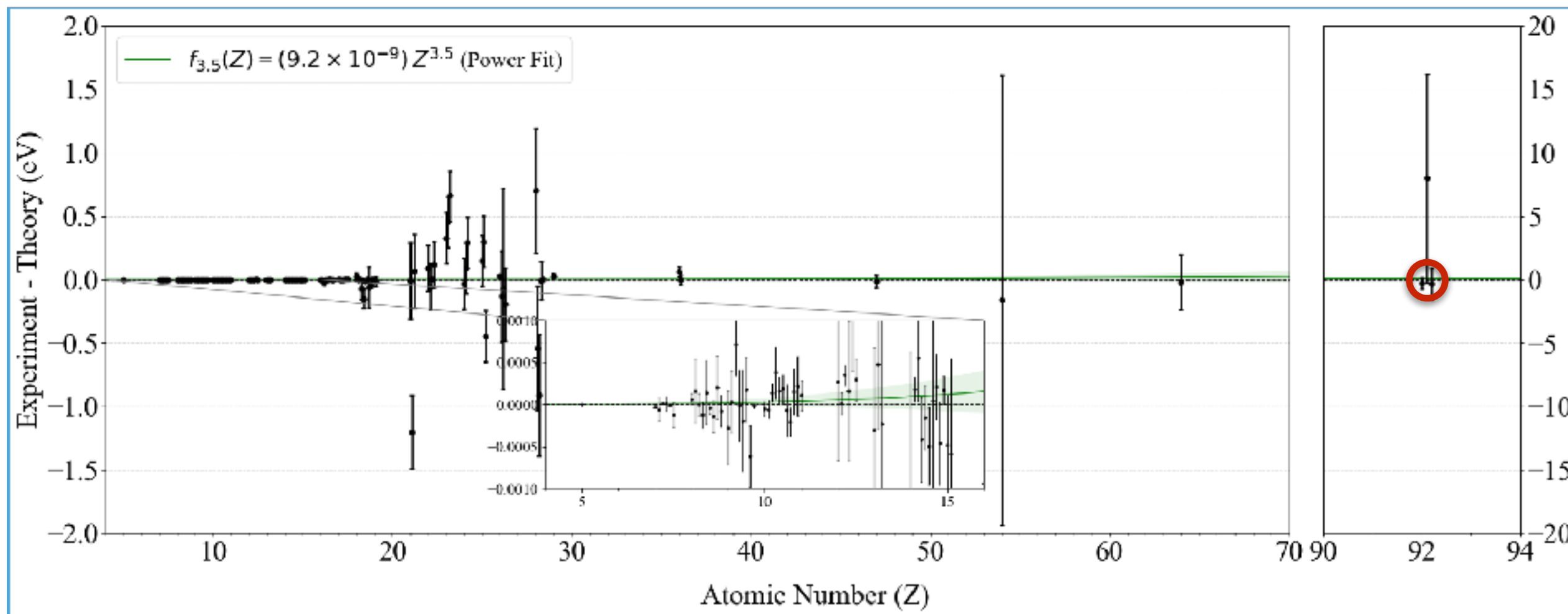
$$n=2 \rightarrow n=1$$



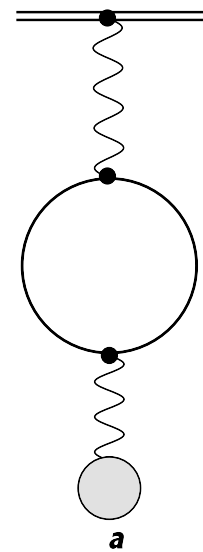
with new measurements (LKB DCS, MPIK Heidelberg, Livermore) reduced deviation
 next improvement: He-like U improved (analysis in progress)

C. Godinho, J. Machado, N. Paul, M. Guerra, P. Indelicato and M. Trassinelli (2023, unpublished)

$$\Delta n=0$$

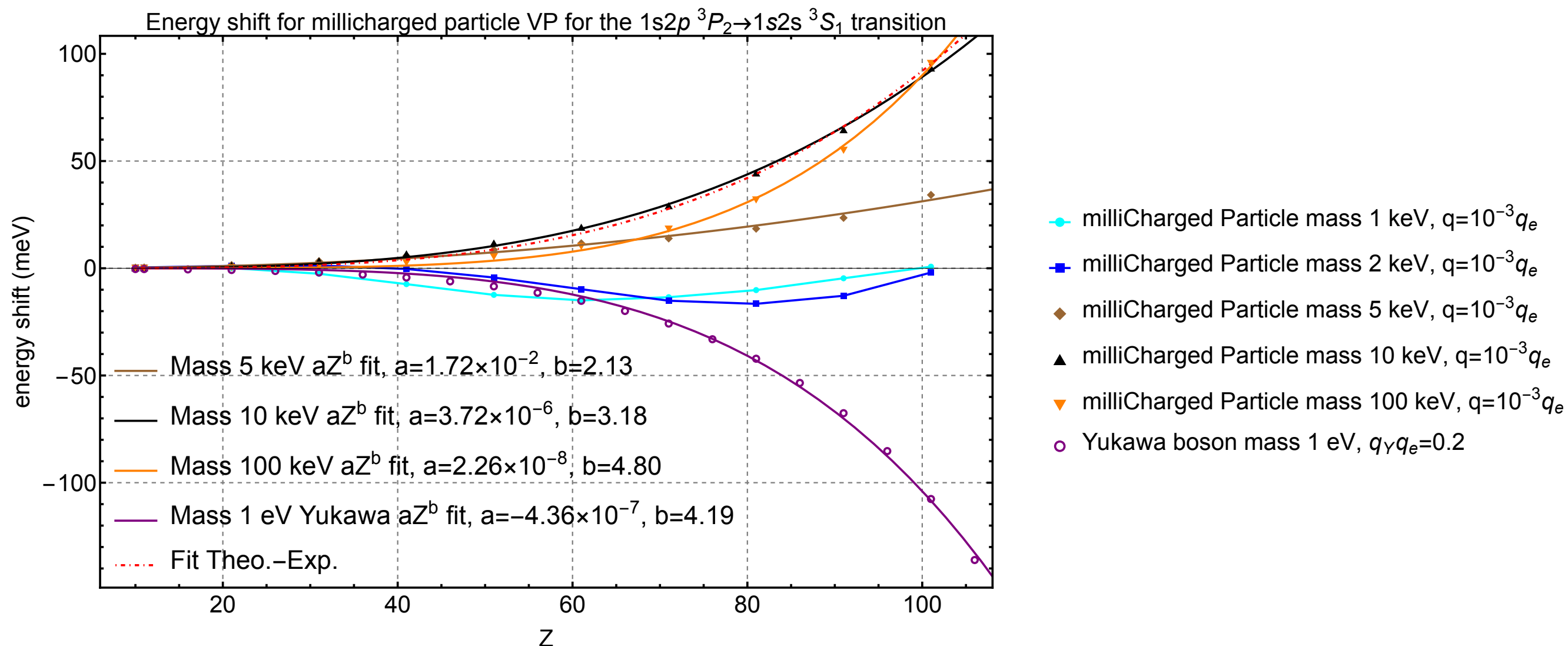


particle of charge: ϵq_e and mass μ



Uehling approximation

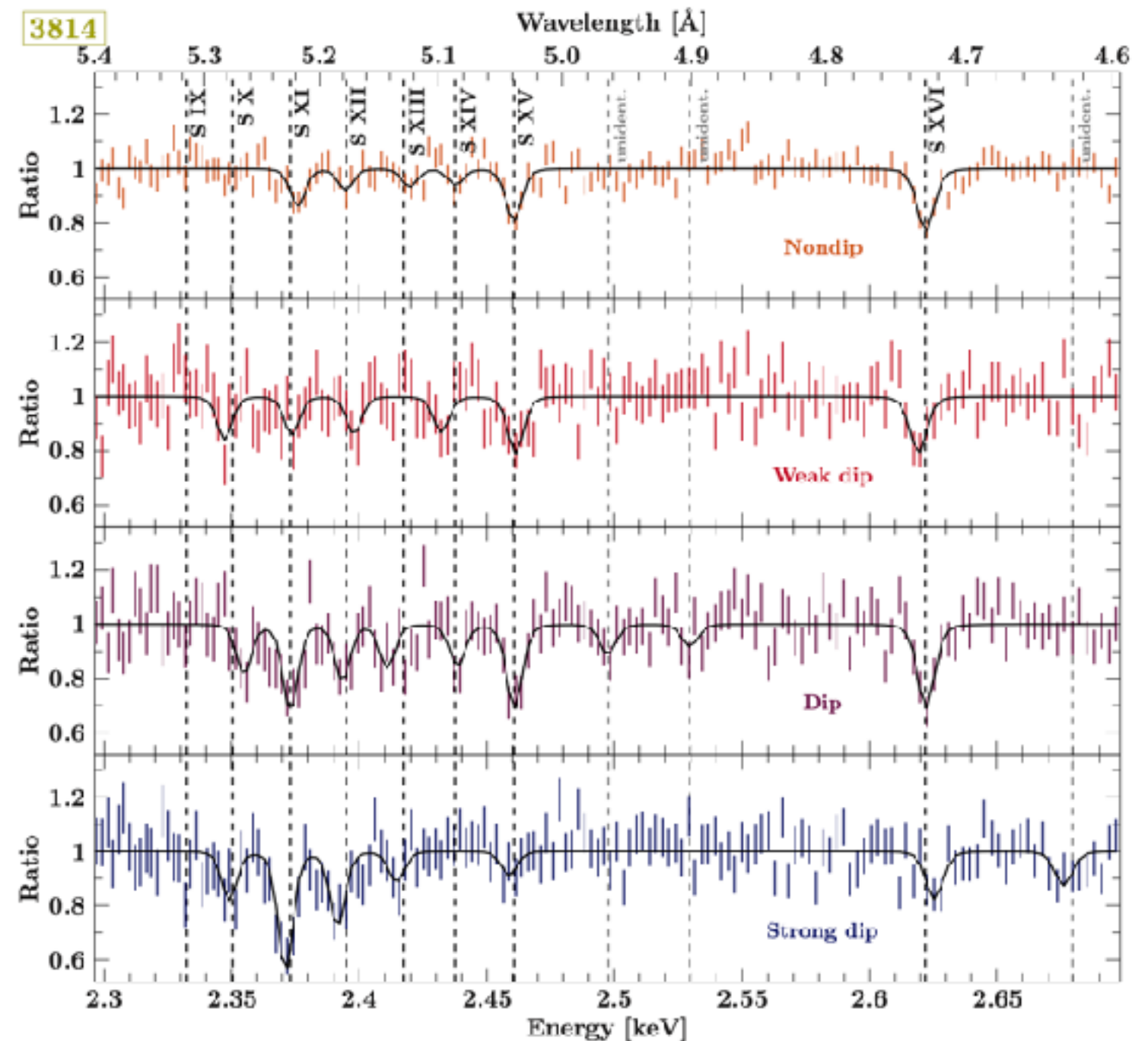
creation of a virtual pair $m_\mu^+ m_\mu^-$

$\Delta n=0$ constraints on dark matter interactions

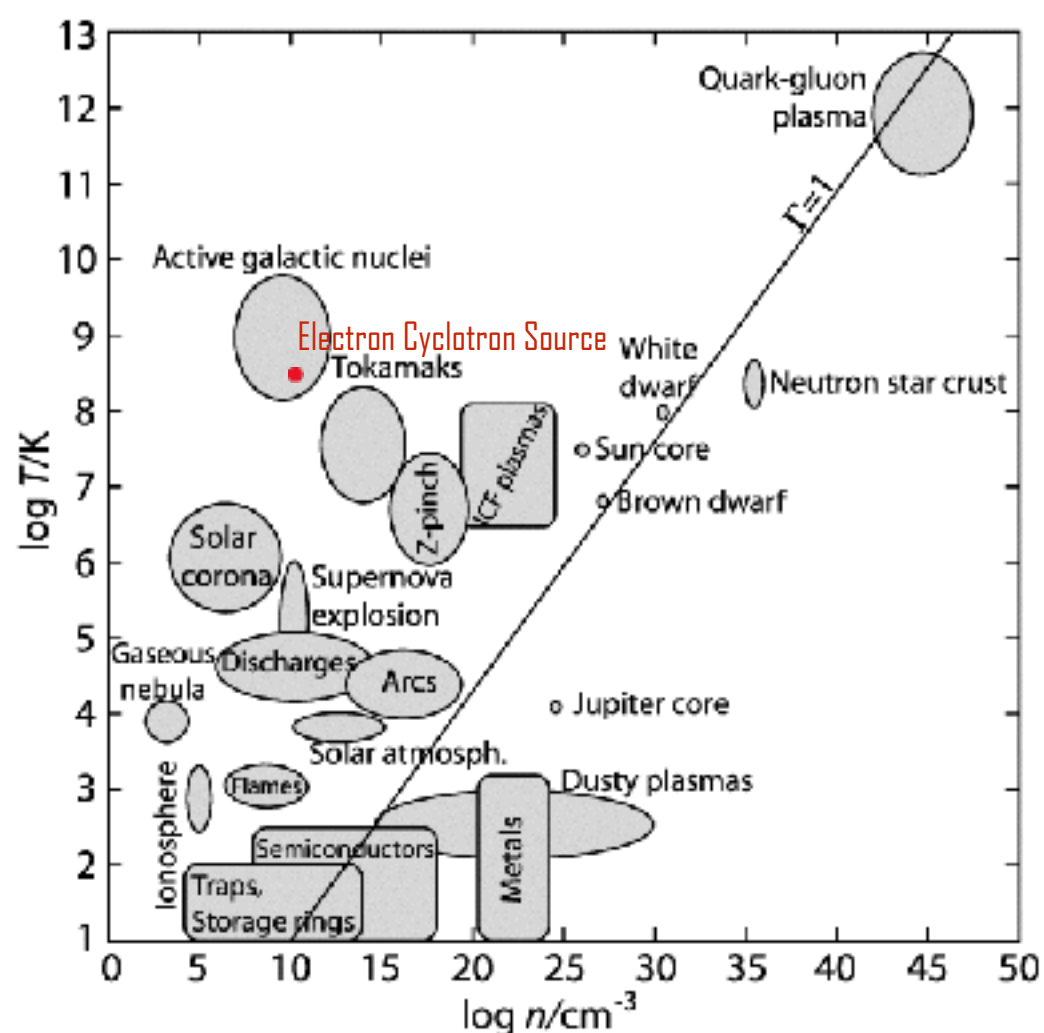
Few-electron ions for astrophysics

Laboratory astrophysics

- Absorption or emission of sulfur
- black hole binary Cyg X-1/ HDE 226868
- Bumps or holes are signatures from inhomogeneities in temperature and densities in the stellar wind powered by the accretion

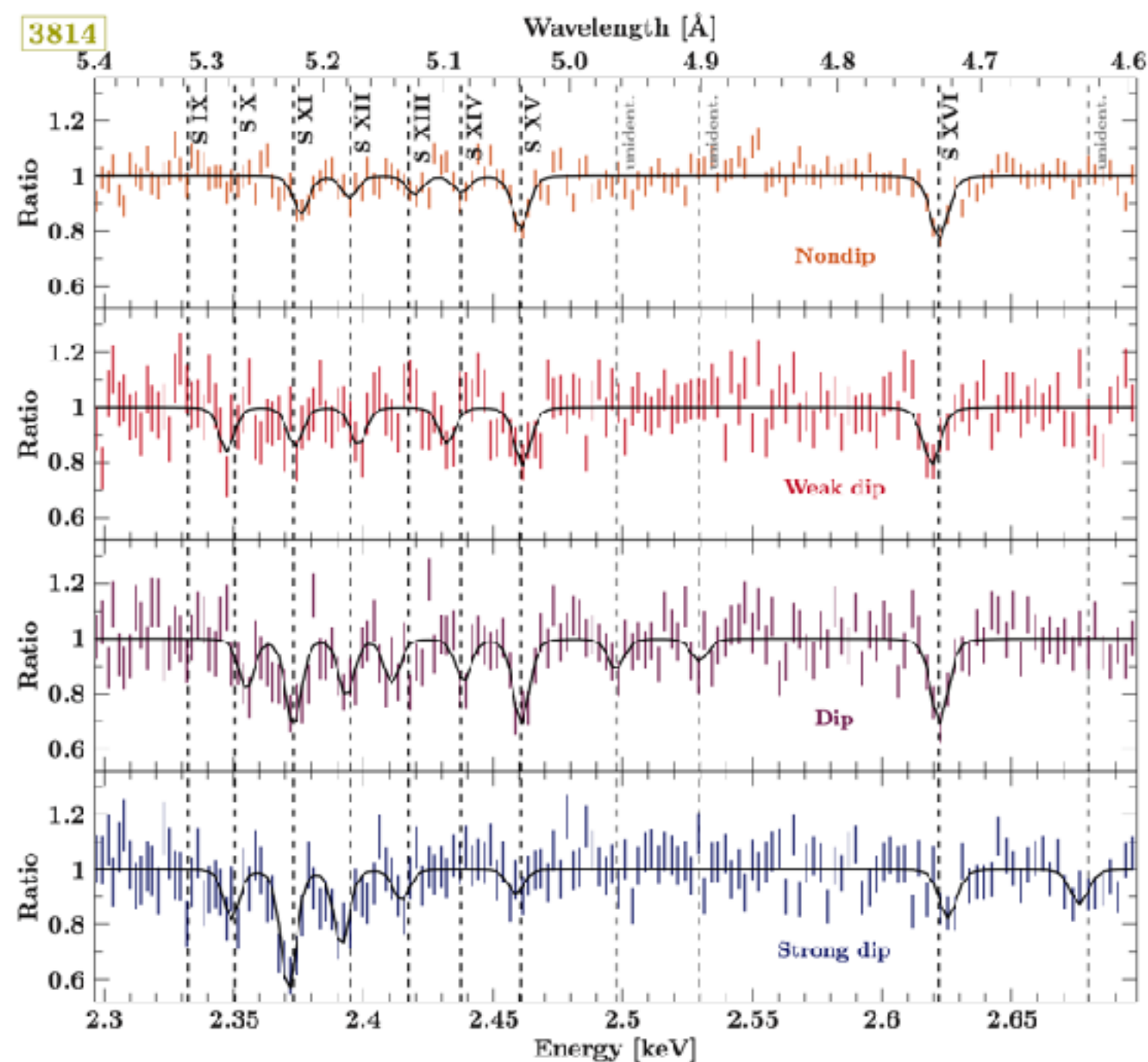


M. Hirsch, N. Hell, V. Grinberg et al, A&A 626, A64 (2019)

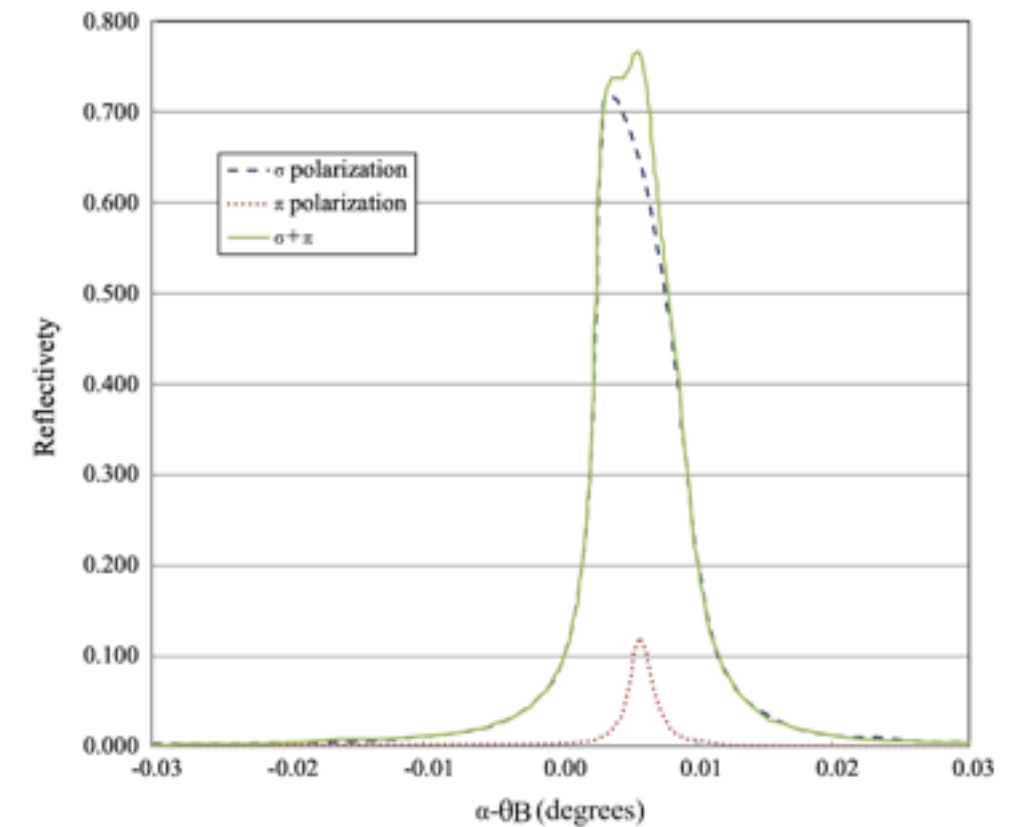
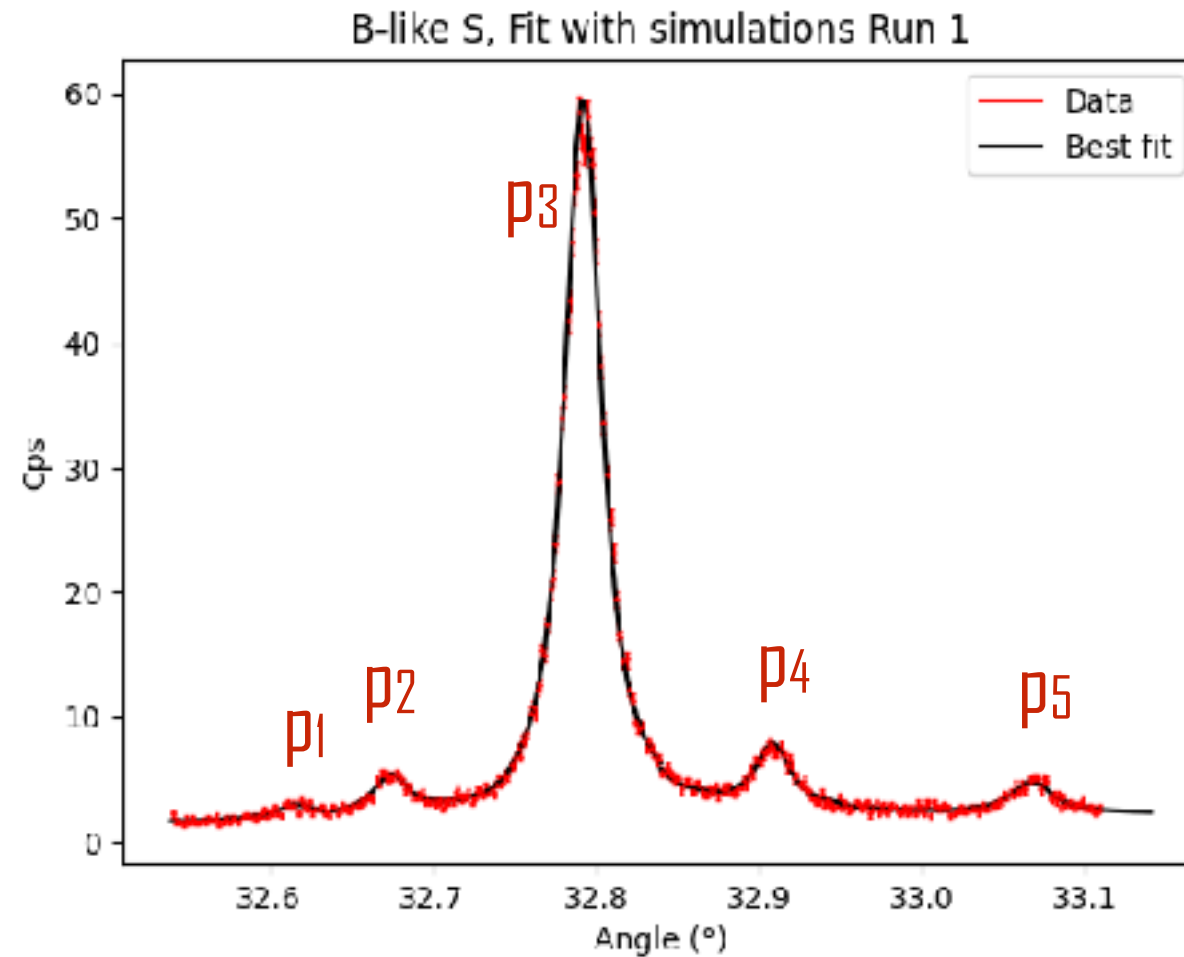


Temperature and electron densities of different plasmas

Donkó, Hartmann and Kalman. (2007). Strongly Coupled Plasma Liquids.



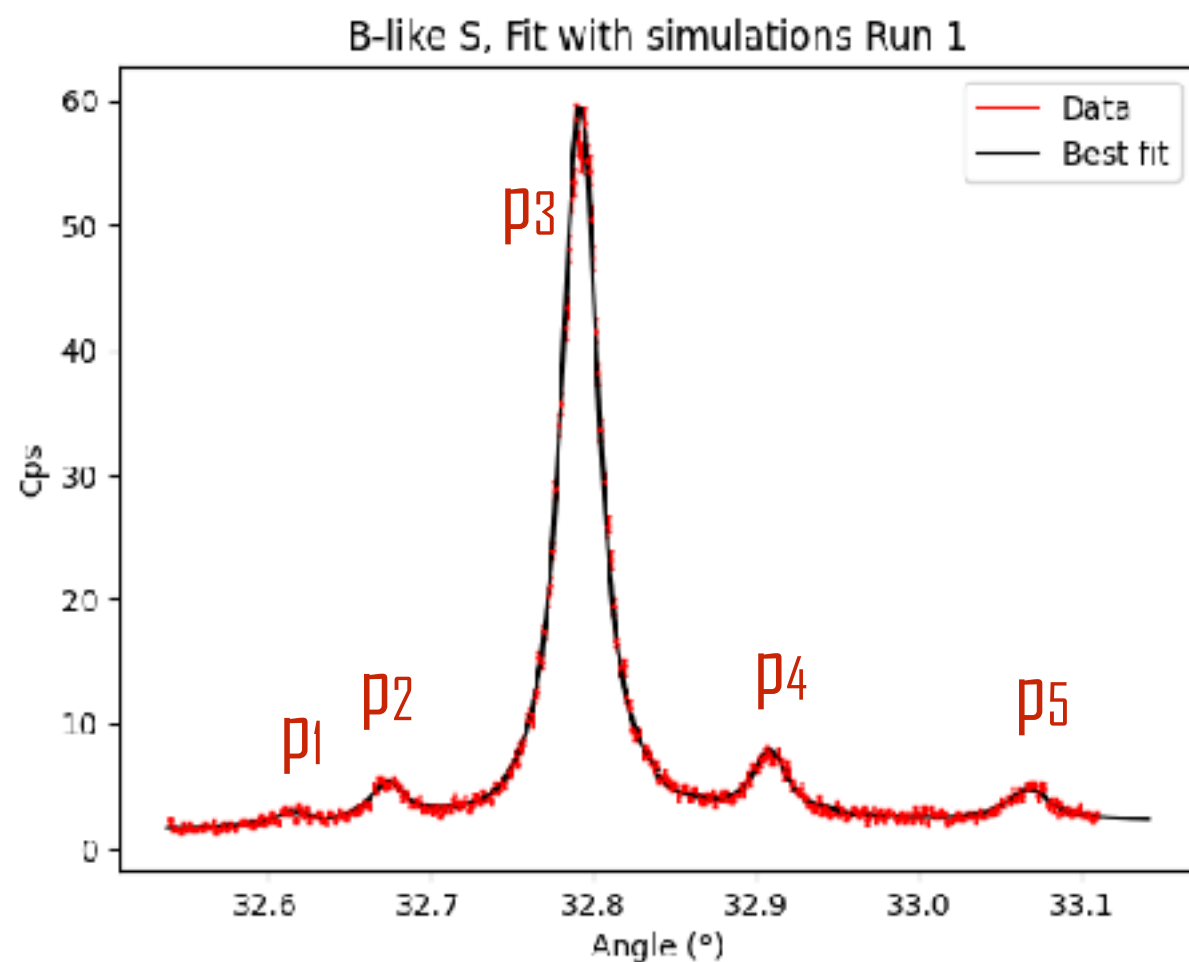
M. Hirsch, N. Hell, V. Grinberg et al, A&A 626, A64 (2019)



Bayesian evidence suggests only a single peak for the large one

Bayesian analysis to determine the number of components in each peak.

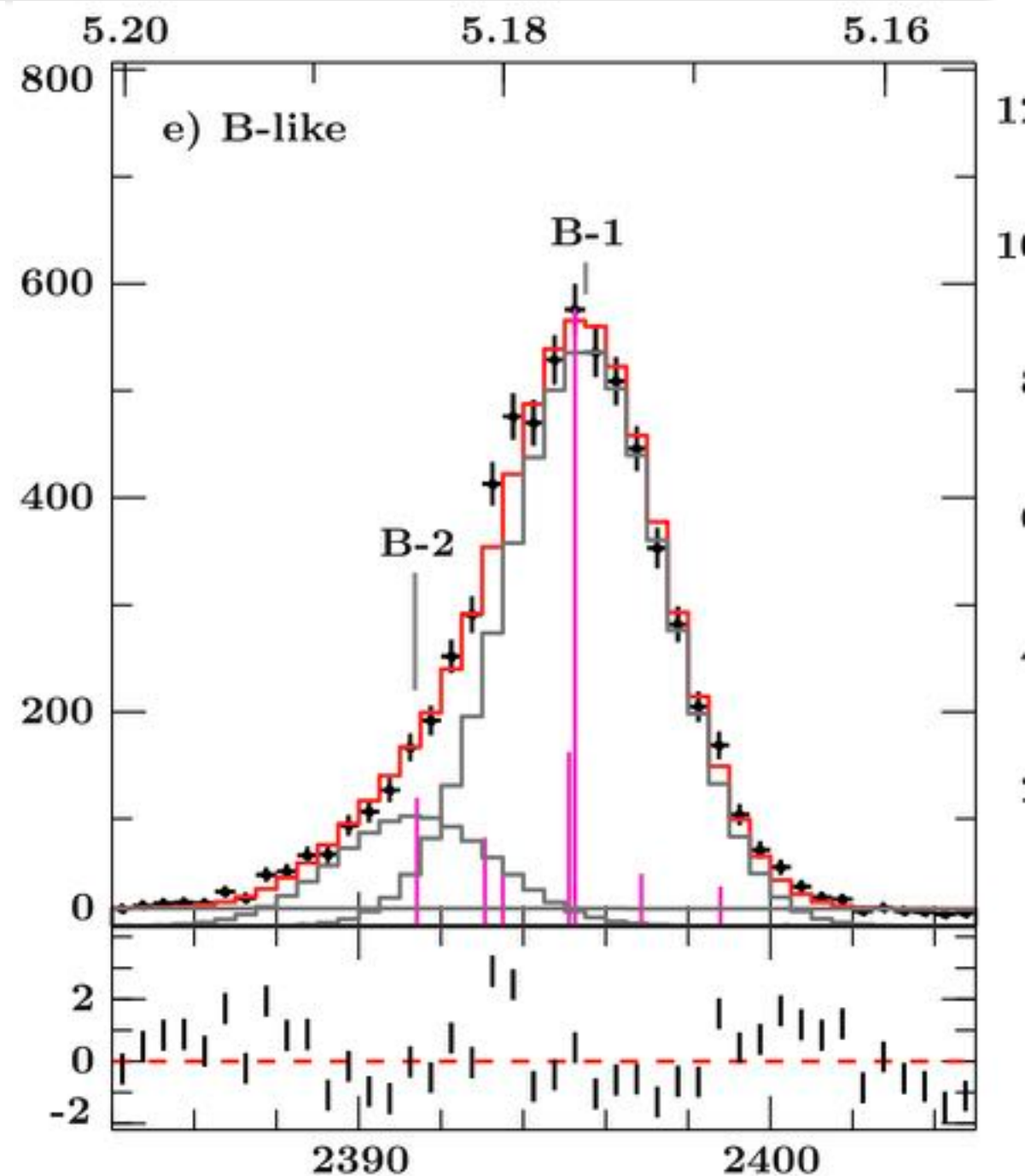
Louis Duval, Jorge Machado, Martino Trassinelli, Nancy Paul, Paul Indelicato



2392.9 eV

2395.5 eV

2399.4 eV



LABORATORY MEASUREMENTS OF THE K-SHELL TRANSITION ENERGIES IN L-SHELL IONS OF SI AND S, N. Hell, G.V. Brown, J. Wilms et al. The Astrophysical Journal 830, 26 (2016). using NASA/GSFC Calorimeter ~ 4.5 eV resolution

Table run per run:

peak	energy (eV)	width (meV)
p1	2392.979(11)	456(35)
p2	2393.778(4)	156(46)
p3	2395.462(5)	182(8)
p4	2397.128(4)	153(25)
p5	2399.405(9)	136(17)

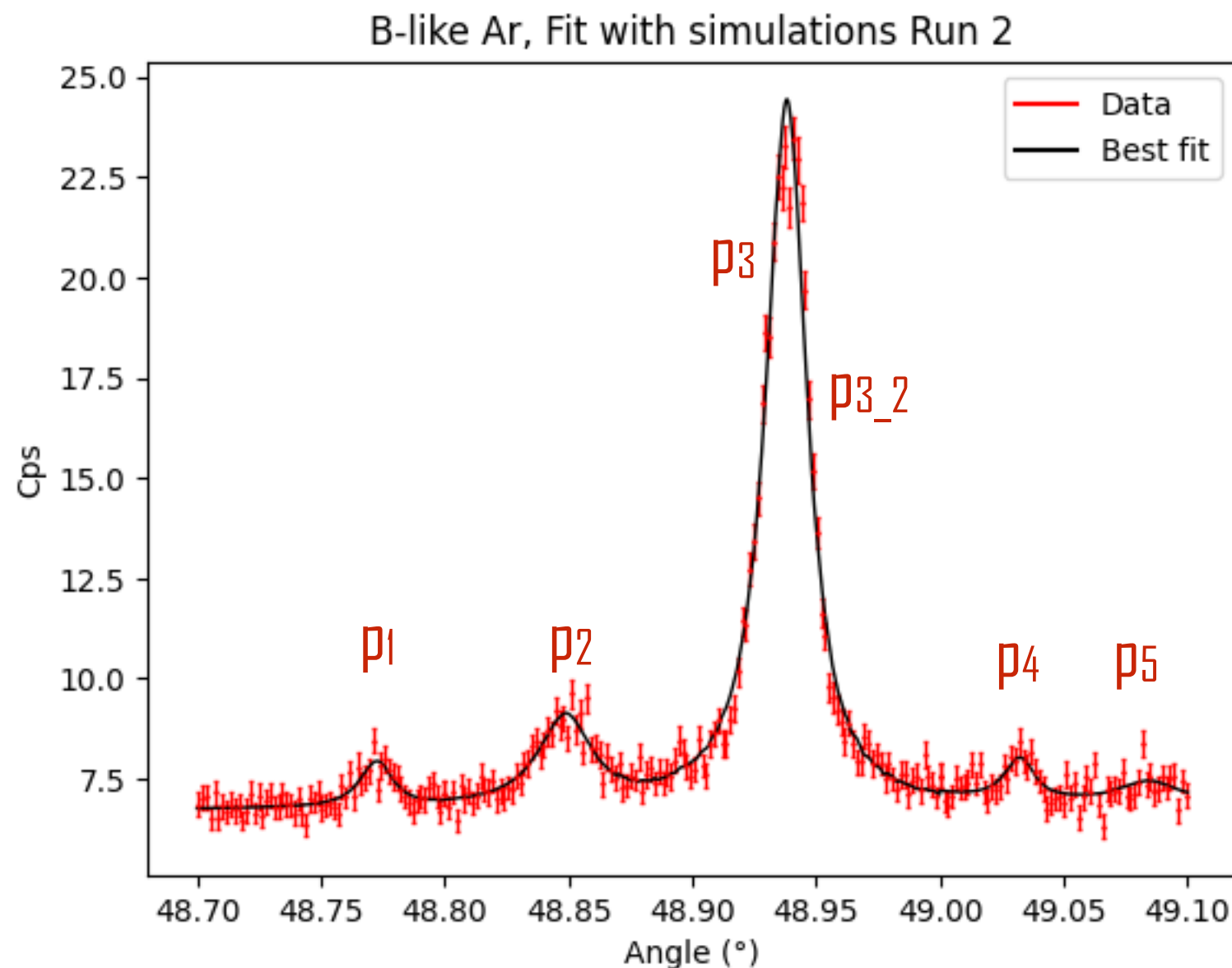
each spectrum analyzed
independently
weighted averaging

Table with sets:

peak	energy (eV)	width (meV)
p1	2392.940(11)	256(35)
p2	2393.778(4)	154(30)
p3	2395.462(5)	192(3)
p4	2397.126(4)	140(17)
p5	2399.405(7)	208(28)

all spectra fitted together

Louis Duval, Jorge Machado, Martino Trassinelli, Nancy Paul, Paul Indelicato



Bayesian evidence
suggests two
lines for the large peak

	peak	energy	sd_energy	amp	sd_amp	gamma	sd_gamma
0	p1	3059.495258	0.052857	0.027862	0.031746	0.104780	0.000005
1	p2	3062.055544	0.025286	0.119467	0.123285	0.351441	0.000008
2	p3	3064.730654	0.010460	0.352798	0.259878	0.241471	0.000002
3	p3_2	3064.999118	0.015824	0.093331	0.224209	0.156352	0.000002
4	p4	3067.783640	0.282822	0.030394	0.117646	0.208913	0.000052
5	p5	3069.349237	0.093624	0.110213	0.146690	0.571780	0.000048

- Example: $1s\ 2s^2\ 2p^2\ ^2P_{1/2}$
 - 40 000 configurations with single, double and triples excitations up to $n=5$
 - 141 Million Coulomb Integrals
 - obliged to neglect non-diagonal Breit integrals
 - new 2TB machine to test larger cases
- Convergence:
 - In many cases convergence is hard or impossible up to now up to even $n=4$, because of the presence of up to 4 simultaneous holes
 - even worse for $1s\ 2s\ 2p^3$ level
- Line identification
 - p5 peak for S not identified yet
 - Ar seems OK within the calculation accuracy

- Role of excitation process in the ECRIS plasma?
 - ions are normally in their ground state
 - B-like level $1s\ 2s^2\ 2p^2$ hole created by $1s$ ionization of C-like ions
 - what about $1s\ 2s\ 2p^3$? ionization and excitation? Ionization+Auger from N-like?

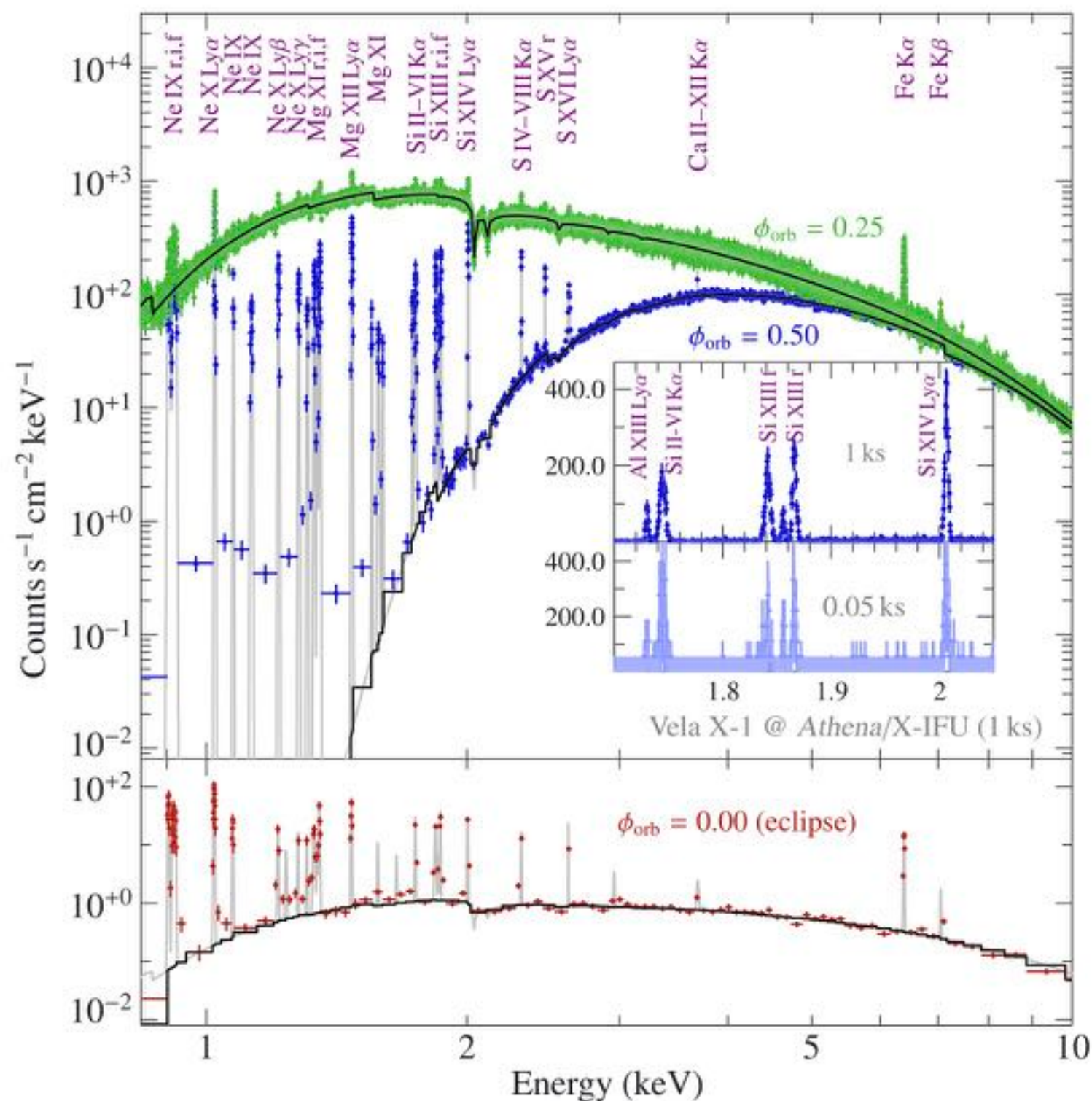


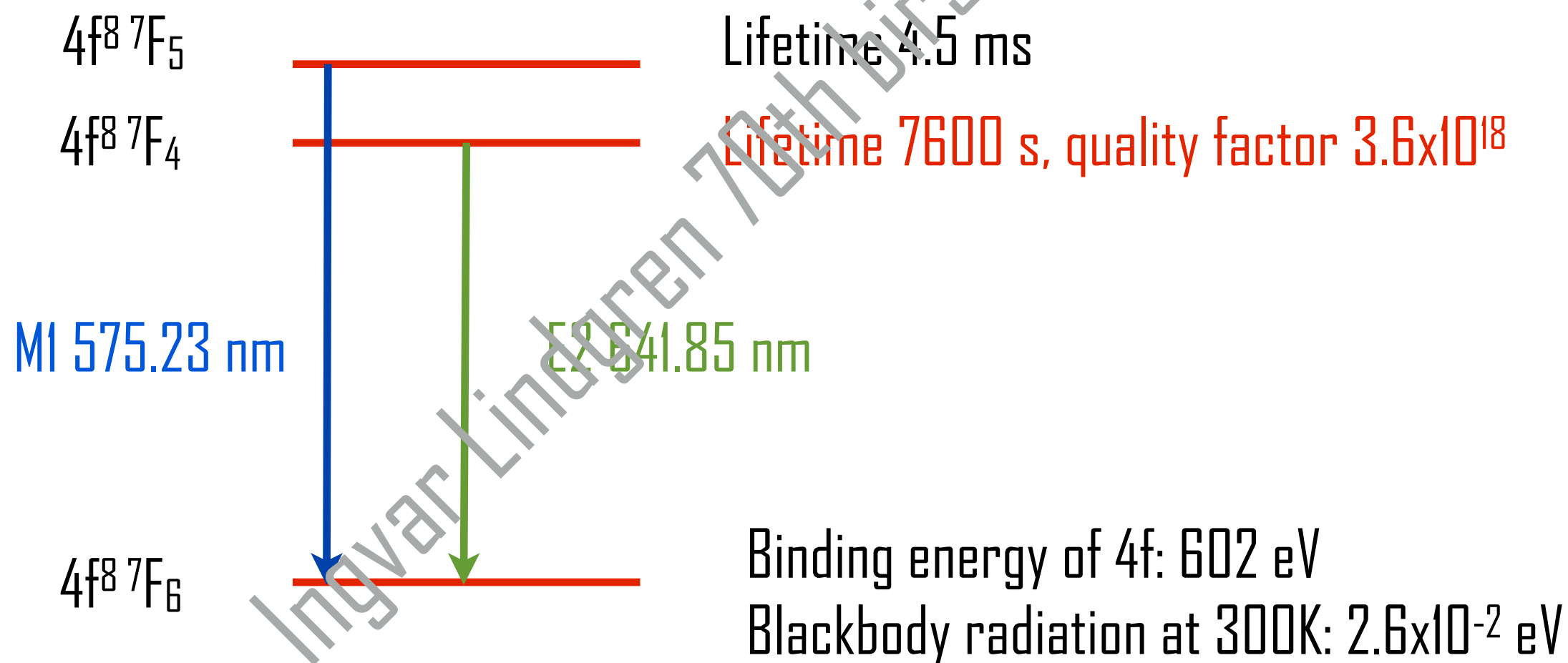
FIGURE 1 Simulations of 1-ks observation with Athena X-IFU of the HMXB Vela X-1 at three orbital phases. The inset compares the Si region of the spectra for 1-ks and 50-s exposures

Highly charged ions in a new era of high resolution X-ray astrophysics, N. Hell, P. Beiersdorfer, G.V. Brown et al. X-Ray Spectrometry 49, 218-233 (2020).

Evaluation of long-lived metastable states

Future high-precision atomic clocks?

- Look for reverse level ordering
 - ★ Ground configuration of Sm-like ions (62 electrons)
 - ★ $^{202}\text{Hg}^{18+}$, m/q : 11.2, close to Be^+ (9)
 - ★ Relativity changes the ground state: $4f^8\ ^7F_6$



Looking at ground state level crossing like 4d-5s ($Z=39, Y^{2+}$), 4f-5s (Nd^{13+})...

J.C. Berengut, V.A. Dzuba, V.V. Flambaum et A. Ong. Phys. Rev. A **86** (2012), p. 022517: “Highly charged ions with E1, M1, and E2 transitions within laser range”

TABLE XI. Scaling dependences for HCIs for various sources of systematic shifts in optical clocks.

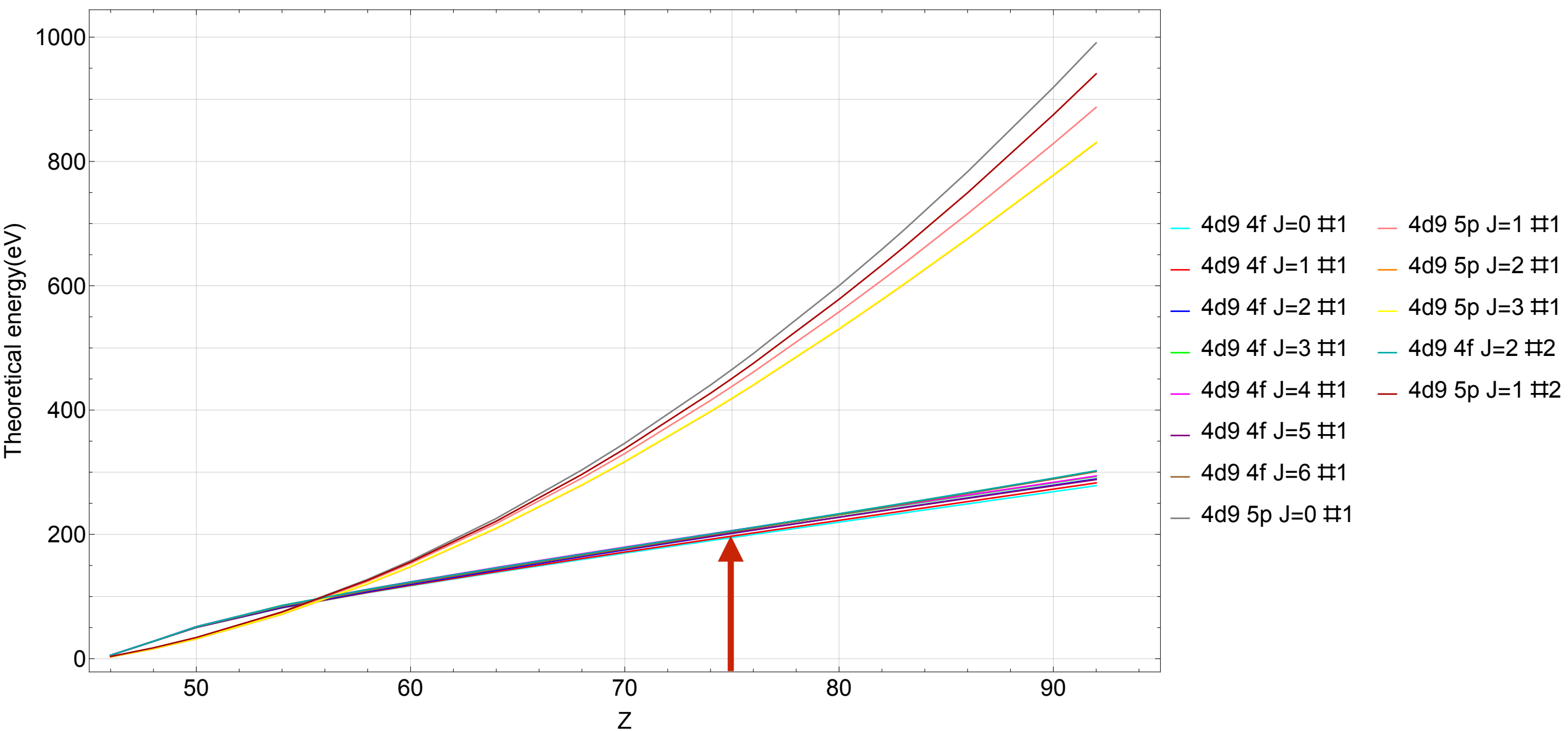
Second-order Stark shift	$\sim 1/Z_a^4$
Blackbody shift	$\sim 1/Z_a^4$
Second-order Zeeman shift	suppressed ^a
Electric quadrupole shift	$\sim 1/Z_a^2$
Fine structure	$\sim Z^2 Z_a^3 / (Z_{\text{ion}} + 1)$
Hyperfine A coefficient	$\sim Z Z_a^3 / (Z_{\text{ion}} + 1)$

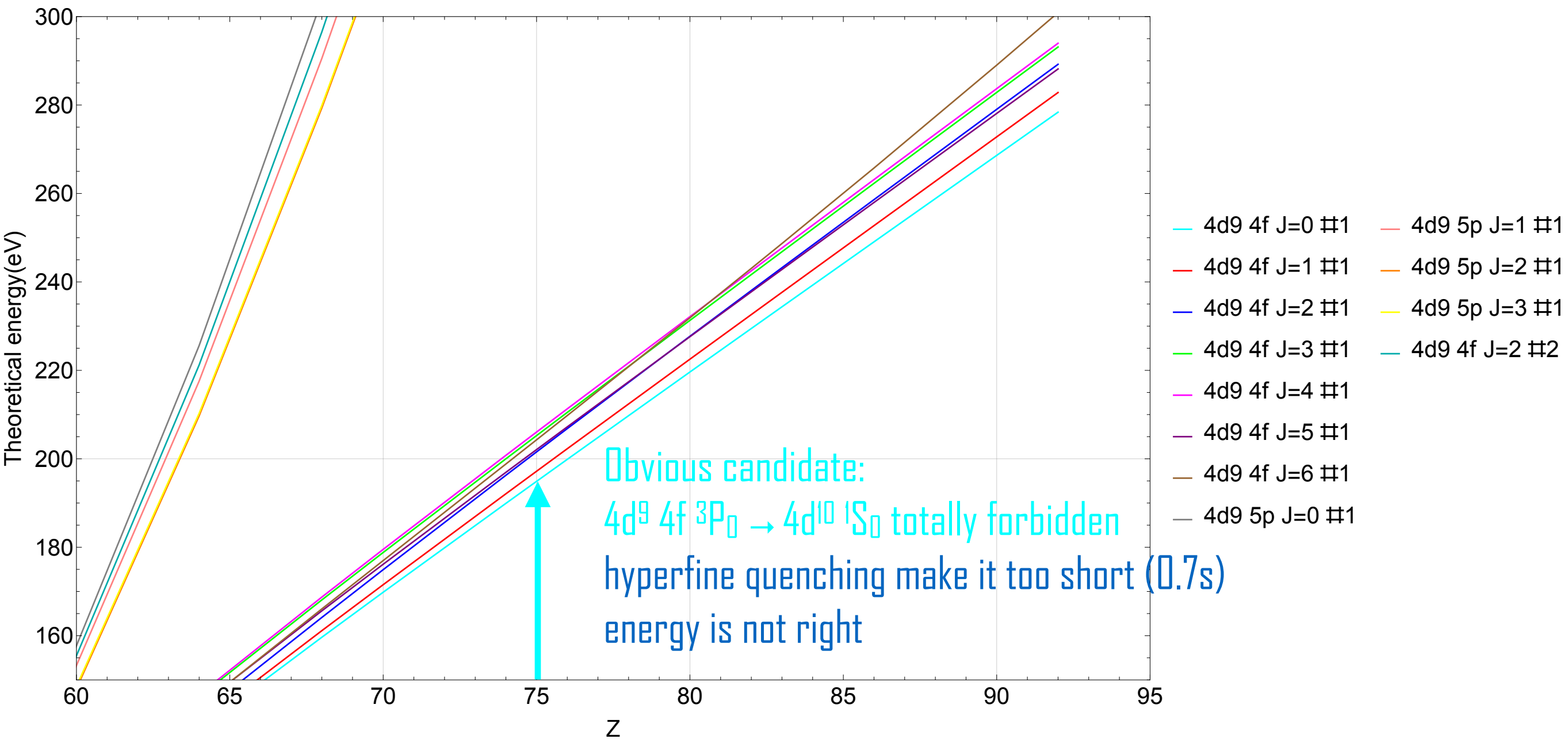
^aThe Zeeman shift is sensitive to the specific fine and hyperfine structure of the transition, but may be suppressed in HCIs due to a larger energy denominator.

- Measurement of the total mass to 1 eV accuracy performed for ^{187}Re ($Z=75$, Pd-like) and ^{187}Os ($Z=76$, Ag-like) performed with PENTATRAP at the MPIK in Heidelberg (for neutrino mass measurement)
- Pd and Pd-like ions have a closed shell ground state $4d^{10} 1S_0$
- During the $^{187}\text{Re}^{29+}$ measurement, it appeared that around 50% of the time the trapped ion had a different mass
- It soon appeared it was not an experimental problem of some sort
- The measurement time in the PENTATRAP is a couple of hours
- Three methods were independently used to look for possible cases of (very long-lived) metastable states without knowing the value of this mass difference

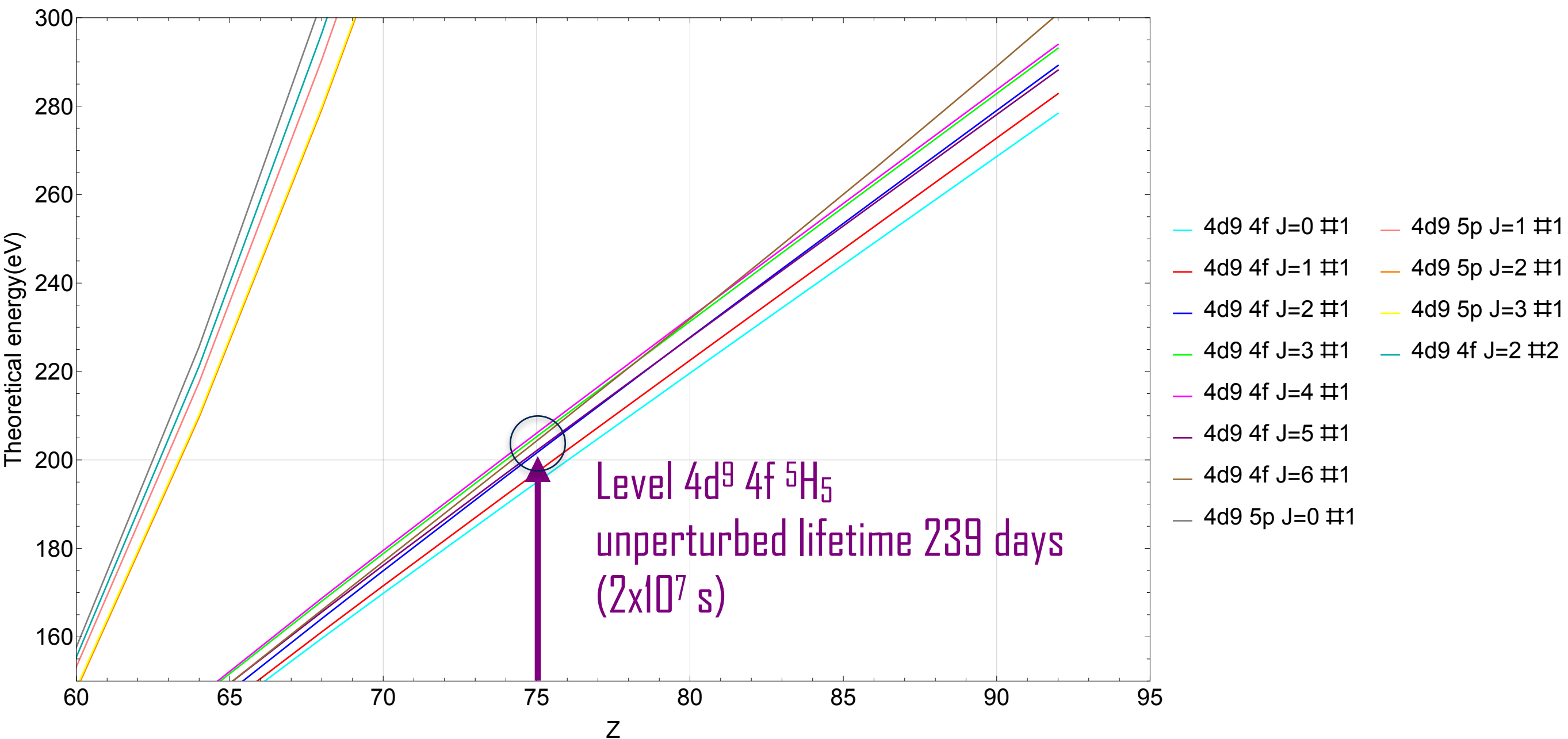
Detection of metastable electronic states by Penning trap mass spectrometry, R.X. Schüssler, H. Bekker, M. Braß et al. Nature 581, 42-46 (2020).

lower level energies, relative to the $4d^{10} \ ^1S_0$ ground state

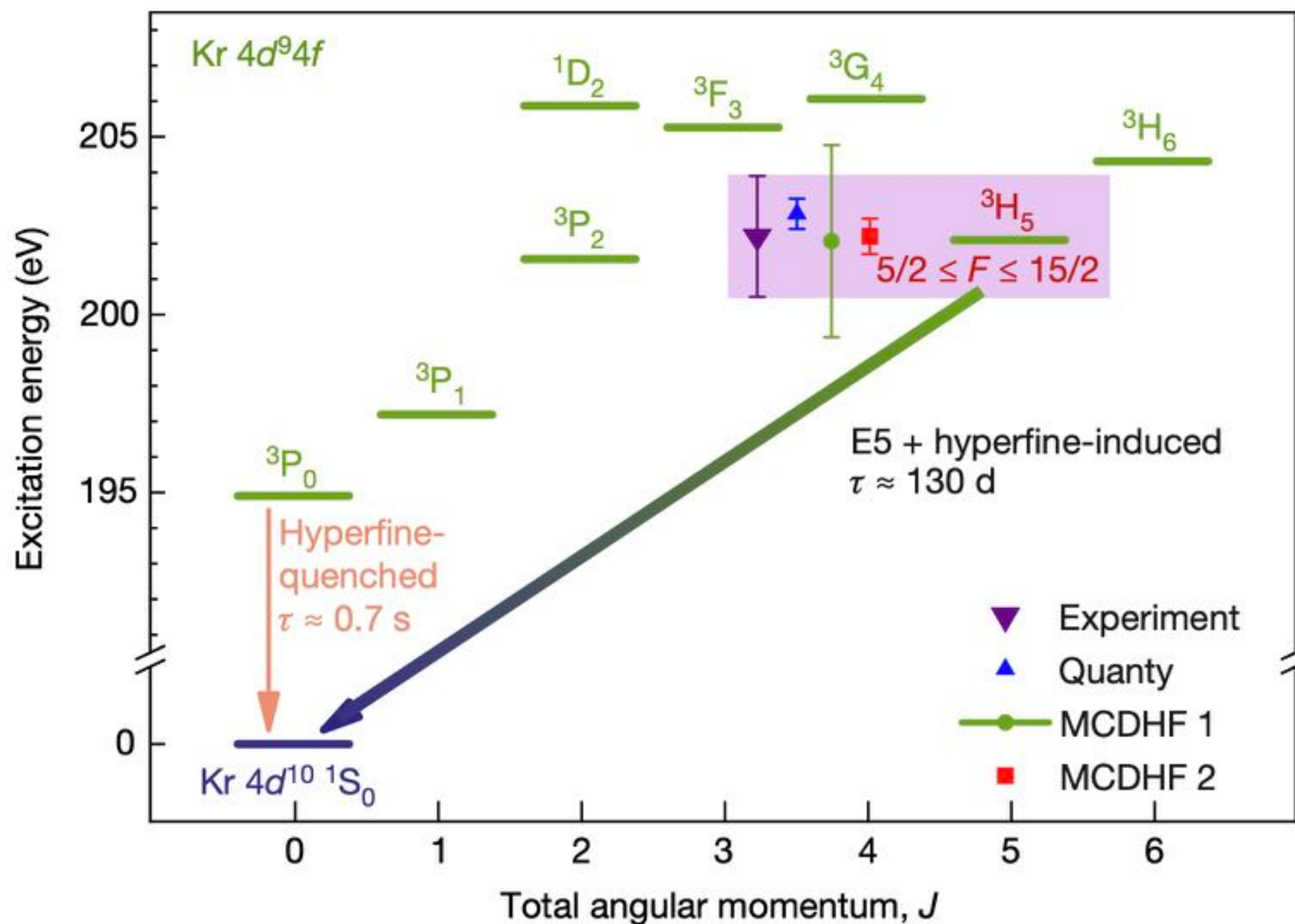


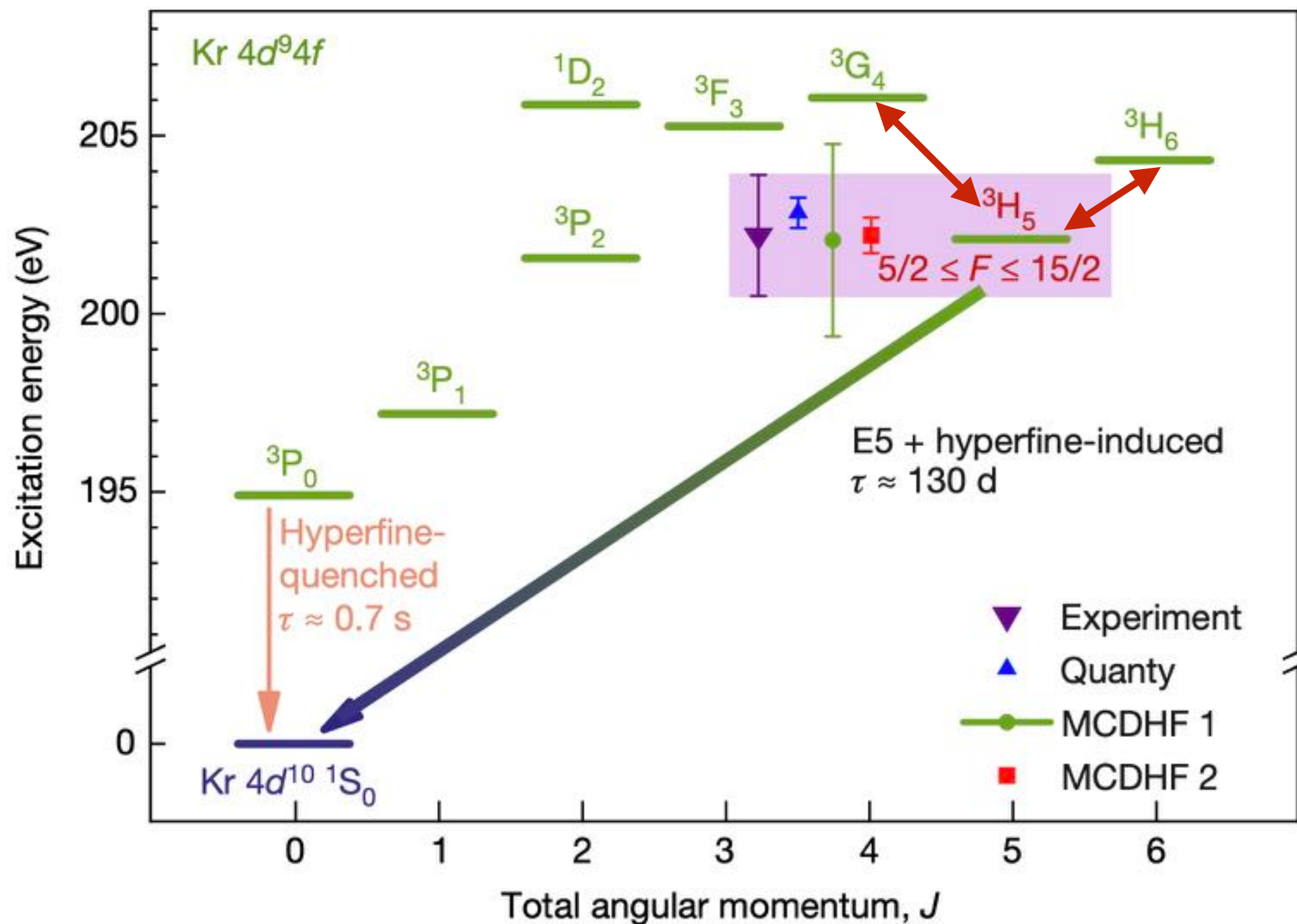


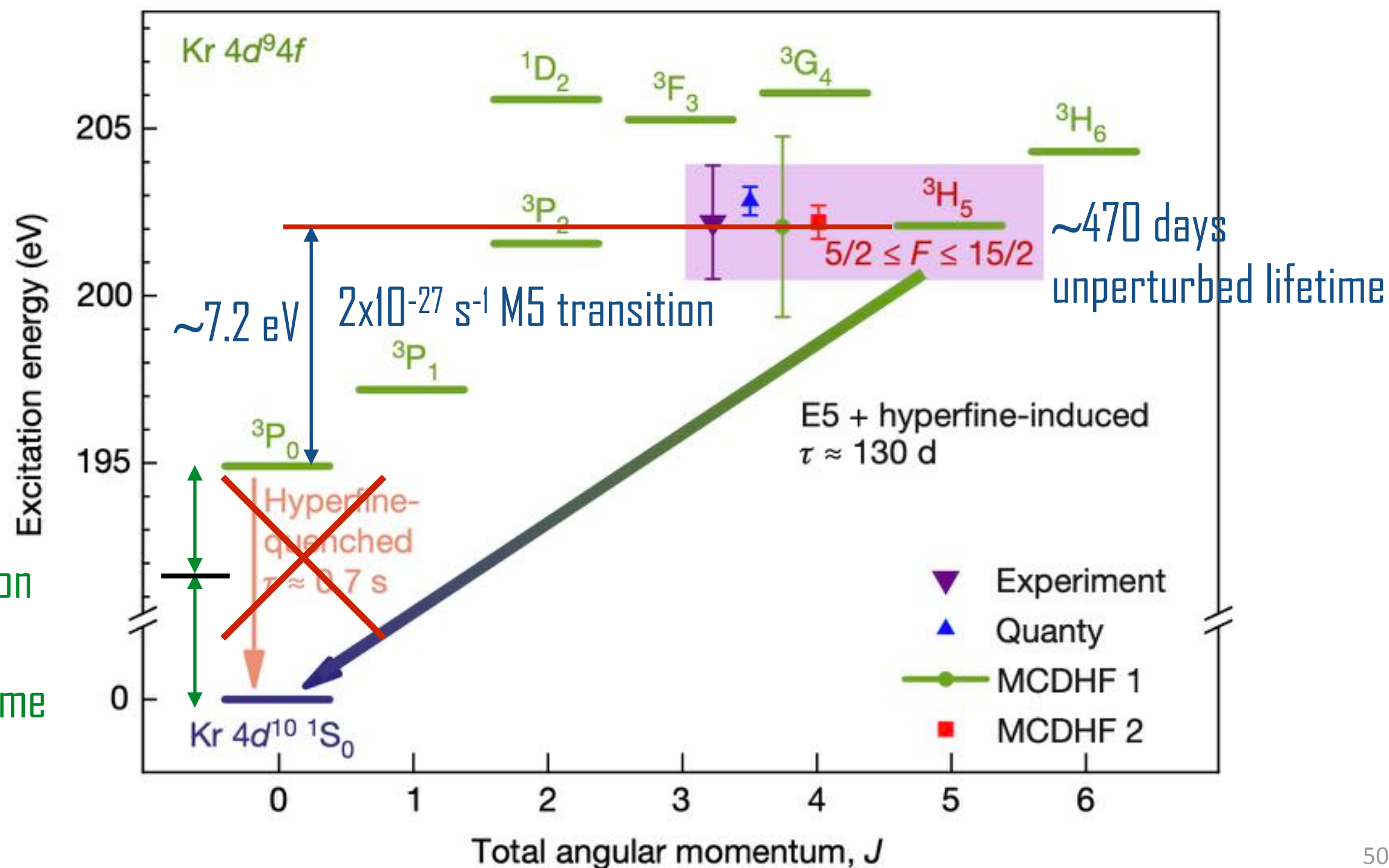
Pd and Pd-like ions have a closed shell ground state $4d^{10} \ ^1S_0$



$4d^9 4f^3 H_5$ unperturbed lifetime **239** days

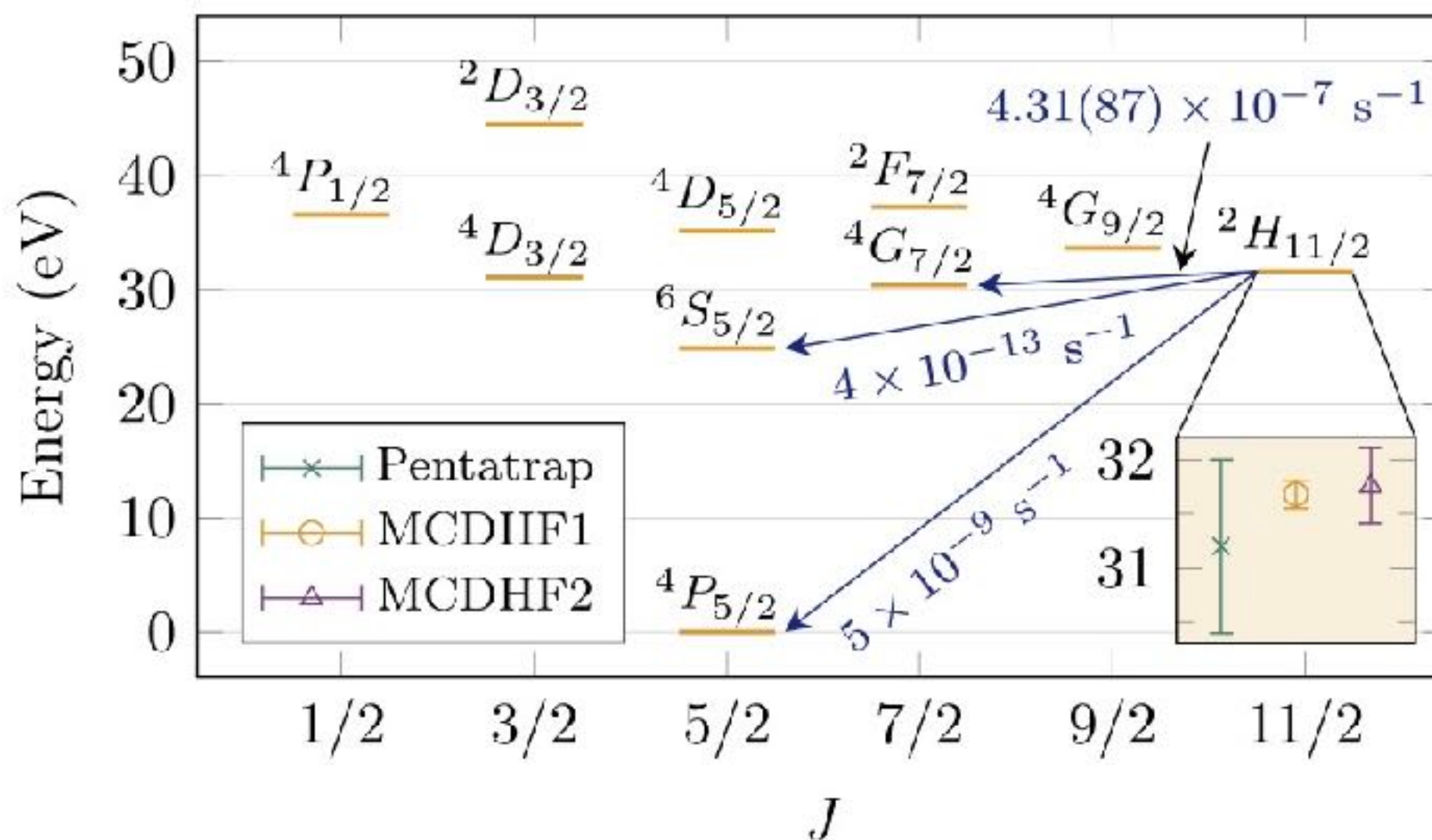


$4d^9 4f^3 H_5$ ^{187}Re has a nuclear spin: $I=5/2$ and a nuclear Q-pole moment

$4d^9 4f^3 H_5$ ^{190}Os ($Z=76$) has a nuclear spin $I=0$ so no hyperfine quenching ^{187}Os : 1.6% abundance, ^{188}Os : 16.1 % abundance, ^{190}Os : 24.6% abundance

- Three different calculations have been performed to check the convergence
 - GRASP 2018* (Zoltan Harman):
 - Layer by layer optimisation
 - excitation of the $n=4$ orbitals up to $9h$
 - $\Delta E = 202.2(5) \text{ eV}$
 - MCDFGME 2020 (PI)
 - smaller set
 - Fully relaxed orbitals
 - Self-consistent magnetic and retardation interaction
 - higher-order QED
 - $\Delta E = 202.1(27) \text{ eV}$
 - QUANTY** (M. Haverkort)
 - multi-Slater-determinant states constructed from relativistic Kohn-Sham single-electron orbitals
 - local spin density functional
 - $\Delta E = 202.41 \text{ eV to } 203.26 \text{ eV}$

- $^{187}\text{Re}^{29+}$: Pd-like ($Z=75$, 46 electrons) $[\text{Kr}]4d^{10} J=0$
- total correlation energy double excitations:
 - from $1s$ to $4f$: 472 configurations
 - from $1s$ to $5g$: 4 892 configurations
 - from $1s$ to $6h$: 17 654 configuration. Why not?
 - cannot index all retardation coefficients $> 2^{31}-1=2 \times 10^9$ integrals = 56 Gb memory, integer overflow
 - going to 64 bit (1.7×10^7 TB!) integers will lead to huge computer time for the large cases and require huge RAM to be efficient
 - cannot hold all the coefficients in memory $\Rightarrow$ calculation duration problem...
- limit to excitations from $4s$, meaning one assumes inner core orbitals below $4s$ are weakly affected by the outer shells



- Experimental progress and astrophysical needs require improved accuracy for transition energies, lifetime, Auger rates...
- Current methods reach their limit pretty fast due to physics (large number of configuration and interaction integrals), methods and computer limitations (RAM, speed...)
- High-accuracy measurements on highly charged ions with more than 3 electrons, leads to the need of accurate theoretical calculations for complicated systems
 - complex outer-shell structure
 - many holes
- Other challenges:
 - laser spectroscopy, in particular of super-heavy elements ($Z \geq 103$)
 - use of long lived-metastable states for Atomic clock insensitive to black body radiation and with high quality factors
 - improve transition energy calculations to manipulate the clock HCI (shelving...)

- Precision x-ray measurements and predictions needed for astrophysics applications, in particular with the new satellites with micro-calorimeters
 - XRISM launched in September
 - New China x-ray satellite (2030)
 - ATHENA with XIFU (>2030)
- Improvement of the Paris DCS for faster setup and change of energy (done)
- MIVOC method for generating metallic ions of astrophysical interest in the SIMPA ion source
- Code developments
 - full parallelisation (complicated gestion of RAM exchange between processors)
 - self-energy screening self-consistent
 - use AI for initial condition (machine learning?) and convergence (neural networks?)
 - global convergence methods like “nested sampling”

Thank you for your attention