華中師範大學
CENTRAL CHINA NORMAL UNIVERSITY

Nuclear Structure Effects to High-Precision Spectroscopy in Hydrogen-Like Atoms

Chen Ji

Institute of Particle Physics
Central China Normal University

International Symposium on Stored Ions for Precision Measurements
Shanghai 2023.11.22-24

Nuclear structures from spectroscopy

- Precision spectroscopy provides abundant information on nuclear structures.

Nuclear structure observables

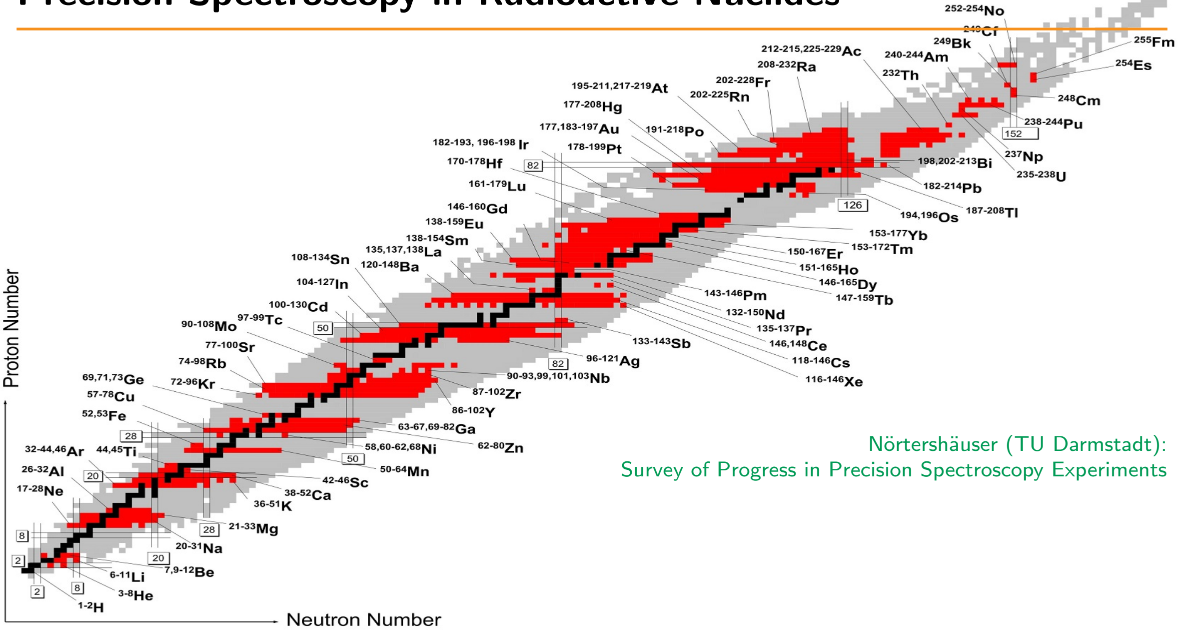
Nuclear spin
Charge radius
Magnetic dipole moment
electric quadrupole moment
magnetic radius

Nuclear structure physics

Nuclear shell evolution
New β stability line, neutron-rich drip line
Halo structure of radioactive nuclei
Internal nucleon distribution
Nuclear deformation

- Precision measurements on nuclear structures provides crucial guidance to building nuclear Hamiltonian models and nuclear many-body theories
  - Deuteron quadrupole moment $\rightarrow$ nuclear tensor force
  - Magnetic moment/radius $\rightarrow$ meson-exchange current
  - Charge radii for $A \geq 3$ systems $\rightarrow$ three-nucleon force

Precision Spectroscopy in Radioactive Nuclides

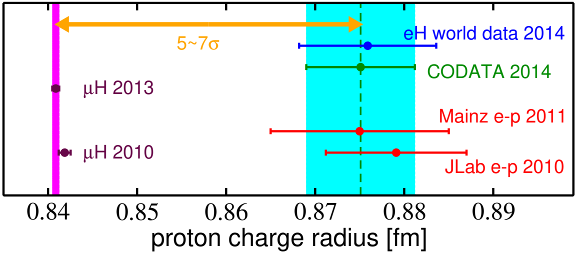


Nörtershäuser (TU Darmstadt):
Survey of Progress in Precision Spectroscopy Experiments

Proton radius puzzle

- electron-proton interaction experiments: $r_p = 0.8770(45)$ fm
  - eH spectroscopy
  - $e{-}p$ scattering
- μ-p interaction experiments: $r_p = 0.8409(4)$ fm
  - μH Lamb shift ($\Delta E_{2\mathrm{S}-2\mathrm{P}}$) [PSI-CREMA]

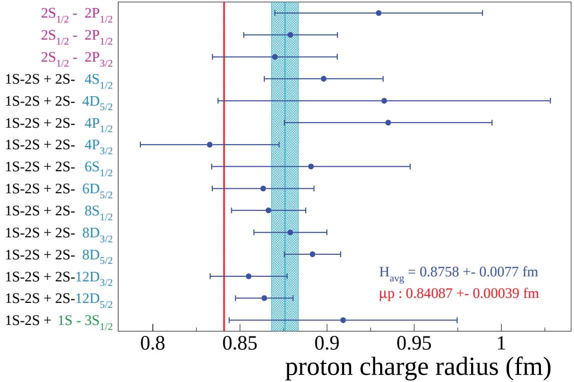
    Pohl *et al.*, Nature (2010); Antognini *et al.*, Science (2013)

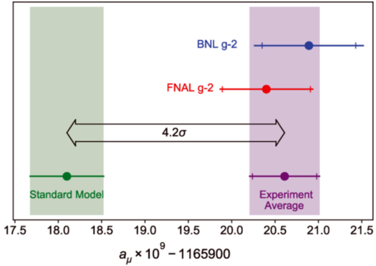


Solve the radius puzzle

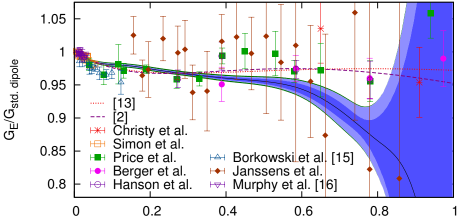
- Possible explanation:
  - Lepton-universality violation? $(g_\mu - 2)$
  - exotic hadron structure?
  - Neglected systematic uncertainty?

No explanation has been completed accepted



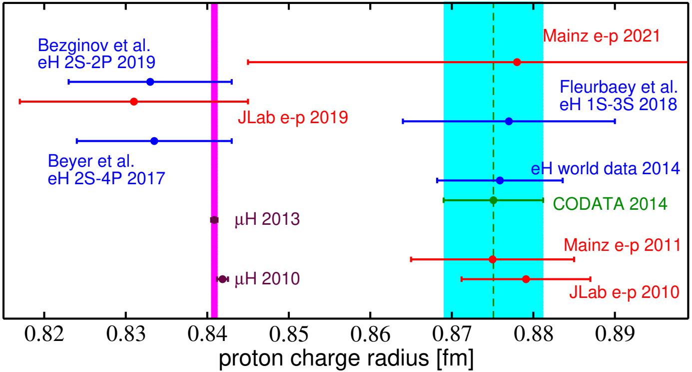


$(g-2)_\mu$ collaboration, PRL 126 (2021) 141801



Solve the radius puzzle

- New experiment to measure the proton radius
  - $e-p$ scattering (JLab, Mainz, Tohoku U.)
  - $\mu-p$ scattering (PSI-MUSE)
  - hydrogen spectroscopy (MPQ, LKB, York U.)



We seem to better (not fully) understand the proton radius now.

Spectroscopy measurement of nuclear radii in other atoms

- Lamb shift in muonic atoms/ions (PSI-CREMA)
  - μ^2H [Pohl *et al.*, Science 2016]
  - μ^4He$^+$ [Krauth *et al.*, Nature 2021]
  - μ^3He$^+$ [K. Schuhmann *et al.*, arXiv:2305.11679]
  - μ^3H, μLi, μBe [planned]

Nuclear charge radii

- $e^{3,4}$He spectroscopy

^{3}He-^{4}He charge-radius isotope-shift

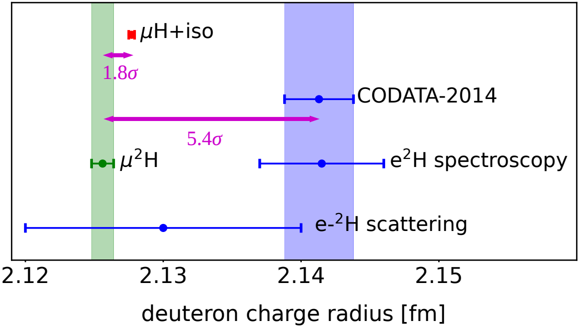
- hyperfine splitting in muonic measurements (PSI-CREMA)
  - μ^2H, μ^3He$^+$ [In plan]

Nuclear magnetic Zemach radius

Deuteron radius puzzle

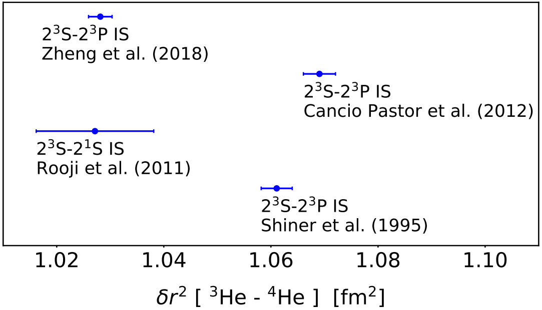
- μ^2H Lamb shift $r_d = 2.12562(78)$ fm   Pohl, *et al.*, Science (2016)
- CODATA-2014: $r_d = 2.1415(45)$ fm

- isotope shift $r_d^2 - r_p^2$:
  $\delta(\mu^2\mathrm{H}, \mu\mathrm{H}) = 3.8112(34)$ fm^2
  $\delta(e^2\mathrm{H}, e\mathrm{H}) = 3.8201(07)$ fm^2 Parthey, *et al.*, PRL (2010)



Charge radii in He isotopes

- Discrepancy in the measurment of $r_E^2(^3\text{He}) - r_E^2(^4\text{He})$



Zheng, et al., Phys. Rev. Lett. 119, 263002 (2017)
Cancio Pastor, et al., Phys. Rev. Lett. 108, 143001 (2012)
van Rooij, et al., Science 333, 196 (2011)
Shiner, et al. Phys. Rev. Lett. 74, 3553 (1995)

Nuclear structure effects to Lamb shift

- Extract nuclear charge radius from Lamb shift in muonic atoms

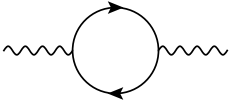
$$\delta E_{\rm LS} = \delta_{\rm QED} + \mathcal{A}_{\rm OPE}\, R_E^2 + \delta_{\rm TPE}$$

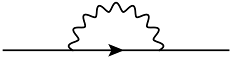
Nuclear structure effects to Lamb shift

- Extract nuclear charge radius from Lamb shift in muonic atoms

$$\delta E_{\rm LS} = {\color{red}\delta_{\rm QED}} + \mathcal{A}_{\rm OPE}\, R_E^2 + \delta_{\rm TPE}$$

- **QED effects**
  - Vacuum polarization (Uehling effect)
  - Lepton self energy
  - relativistic recoil





Nuclear structure effects to Lamb shift

- Extract nuclear charge radius from Lamb shift in muonic atoms

$$\delta E_{\rm LS} = \delta_{\rm QED} + \mathcal{A}_{\rm OPE}\, R_E^2 + \delta_{\rm TPE}$$

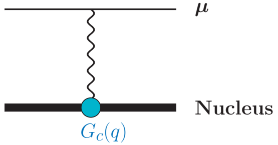
- **Nuclear structure effects**

Nuclear structure effects to Lamb shift

- Extract nuclear charge radius from Lamb shift in muonic atoms

$$\delta E_{\rm LS} = \delta_{\rm QED} + \boxed{\mathcal{A}_{\rm OPE}\, R_E^2} + \delta_{\rm TPE}$$

- **Nuclear structure effects**
  - $\propto R_E^2 \Longrightarrow$ **one-photon exchange (OPE)**
    $\mathcal{A}_{\rm OPE} \approx m_\mu^3 (Z\alpha)^4/12$

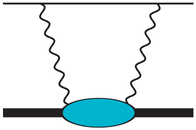


Nuclear structure effects to Lamb shift

- Extract nuclear charge radius from Lamb shift in muonic atoms

$$\delta E_{\mathrm{LS}} = \delta_{\mathrm{QED}} + \mathcal{A}_{\mathrm{OPE}}\, R_E^2 + \boxed{\delta_{\mathrm{TPE}}}$$

- **Nuclear structure effects**
  - $\boldsymbol{\delta_{\mathrm{TPE}}} \Longrightarrow$ **two-photon exchange (TPE)**
    - elastic part: Zemach moment δ_{Zem}
    - inelastic part: nuclear polarizability δ_{pol}

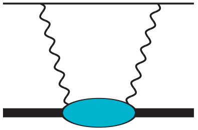


Nuclear structure effects to Lamb shift

- Extract nuclear charge radius from Lamb shift in muonic atoms

$$\delta E_{\rm LS} = \delta_{\rm QED} + \mathcal{A}_{\rm OPE}\, R_E^2 + \boxed{\delta_{\rm TPE}}$$

- **Nuclear structure effects**
  - $\boldsymbol{\delta_{\rm TPE}} \Longrightarrow$ **two-photon exchange (TPE)**
    - elastic part: Zemach moment $\delta_{\rm Zem}$
    - inelastic part: nuclear polarizability $\delta_{\rm pol}$

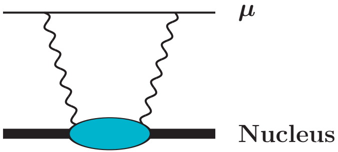


- **The accuracy of extracting R_E relies on the theoretical input of $\delta_{\rm TPE}$**

  μ^2H experiment: $\delta_{\rm pol}$ requires 1% accruacy
  $\mu^{3,4}$He$^+$ experiment: $\delta_{\rm pol}$ requires 5% accuracy

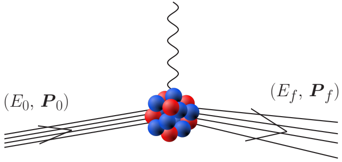
Nuclear polarizability from sum rules for photo-nuclear reactions

$$\delta_{\text{pol}} = \sum_{g, S_{\hat{O}}} \int_{\omega_{th}}^{\infty} d\omega \, \underbrace{g(\omega)}_{\text{weight}} \, \underbrace{S_{\hat{O}}(\omega)}_{\text{response function}}$$



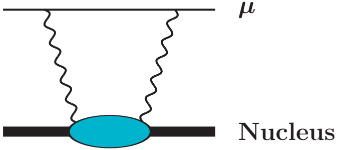
- energy-weighted sum rules $g(\omega)$
- nuclear response function $S_{\hat{O}}(\omega)$

$$S_O(\omega) = \sum\!\!\!\!\!\!\int_f |\langle \psi_f | \hat{O} | \psi_0 \rangle|^2 \delta(E_f - E_0 - \omega)$$



Nuclear polarizability from sum rules for photo-nuclear reactions

$$\delta_{\rm pol} = \sum_{g,\, S_{\hat{O}}} \int_{\omega_{th}}^{\infty} d\omega \; \underbrace{g\,(\omega)}_{\text{weight}} \; \underbrace{S_{\hat{O}}\,(\omega)}_{\text{response function}}$$



Contributing terms in nuclear polarizability $\delta_{\rm pol}$:

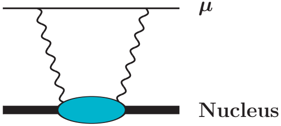
- multipole expansion to EM operators
  - E0, E1, E2, M1 response sum rules
- relativistic and Coulomb-distortion corrections
- intrinsic nucleon structure corrections

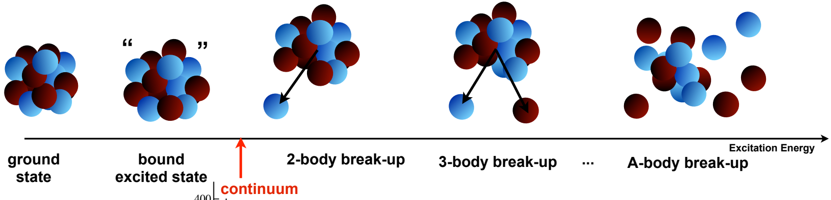
CJ, Bacca, Barnea, Hernandez, Nevo-Dinur, JPG 45 (2018) 093002

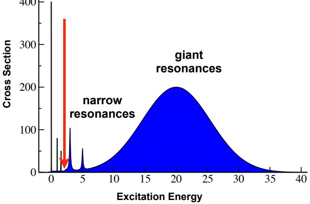
Nuclear response function: continuum spectrum

- The nucleus is virtually excited during the TPE process

$$S_O(\omega) = \sum_f |\langle \psi_f | \hat{O} | \psi_0 \rangle|^2 \delta(E_f - E_0 - \omega)$$





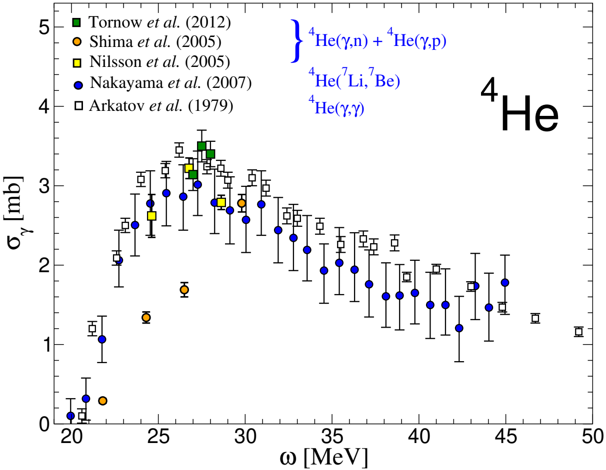


$$\sigma \propto \left| \langle \Psi_f | J^\mu | \Psi_0 \rangle \right|^2$$

Exact knowledge limited in energy and mass number

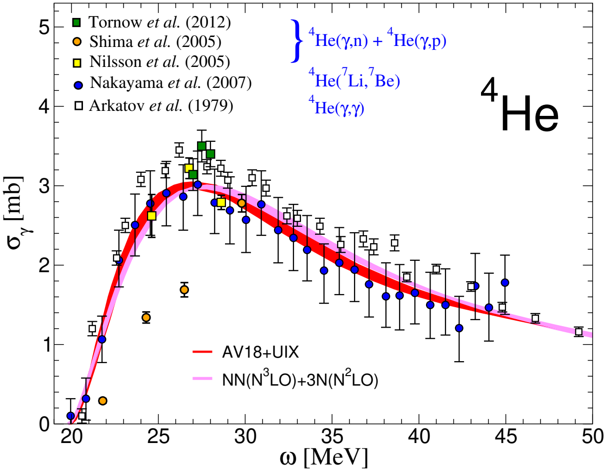
Determine $S_{\hat{O}}$ from photo-nuclear reaction experiments

$$\sigma_\gamma(\omega) = 4\pi^2 \alpha \omega S_{E1}(\omega)$$



Determine $S_{\hat{O}}$ from photo-nuclear reaction experiments

$$\sigma_\gamma(\omega) = 4\pi^2 \alpha \omega S_{E1}(\omega)$$



Ab initio nuclear-structure calculations

Recent developments in nuclear structure theories:

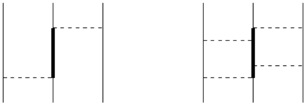
- Microscopic theories of nucleon-nucleon interactions
- Theory for electroweak interactions involving nucleons
- Ab initio calculations of nuclear structures and reactions
- Uncertainty quantification for nuclear theory

Realistic nuclear potential

- **Argonne v_{18} NN interaction**
  - fit to 1787 pp & 2514 np scattering data ($E_{lab} \leq 350$ MeV, $\chi^2/\mathrm{datum} = 1.1$)
  - nn scattering length & deuteron binding energy

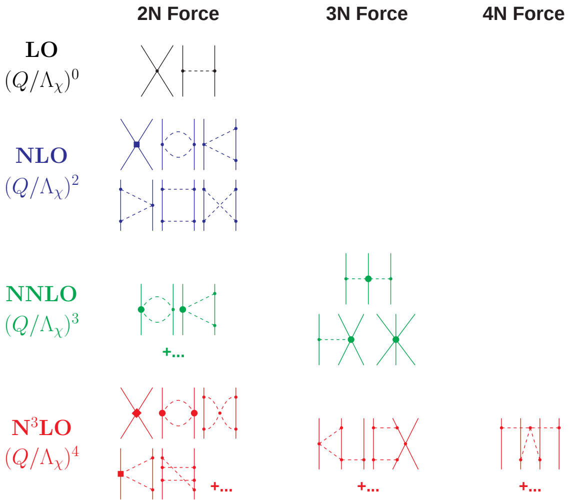
- **Urbana IX NNN force**

$$V_{ijk} = V_{ijk}^{2\pi P} + V_{ijk}^{R}$$



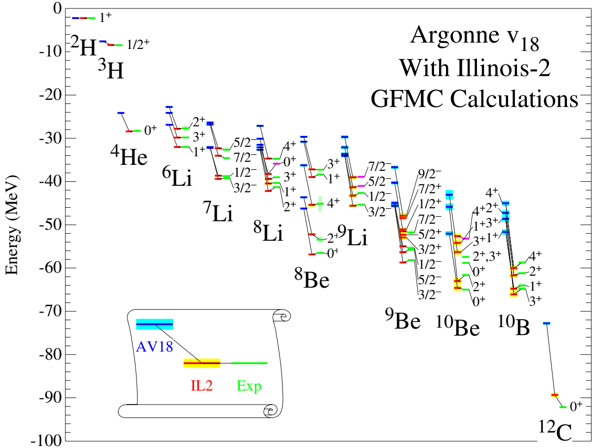
Chiral effective field theory (χEFT)

- The effective theory of QCD at low energies
- Nuclear potential based on power-counting (Q/Λ_χ expansion)
- low-energy constants determined from scattering data

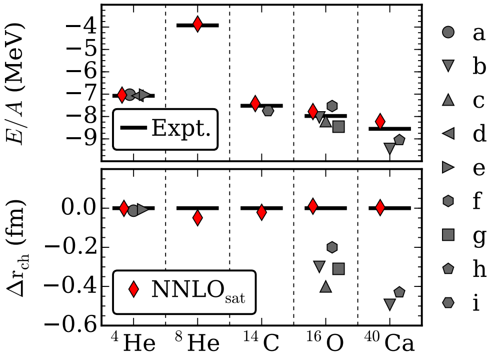


Nuclear ab inito calculations

Combined with quantum many-body calculations, state-of-the-art nuclear forces can accurately describe a wide range of nuclear structures



AV18+UIX Carlson al., RMP 87, 1067 (2015)



χEFT

Ekström et al., PRC 91, 051301 (2015)

Ab-initio calculations of nuclear polarizability $\delta_{\rm pol}$

- $\mu^{2,3}$H, $\mu^{3,4}$He$^+$:
  - Numerical ab-initio methods

    Effective Interaction Hyperspherical Harmonics Expansion
    Lorentz Integral Transform (response function)
    Lanczos Algorithm (sum rules)

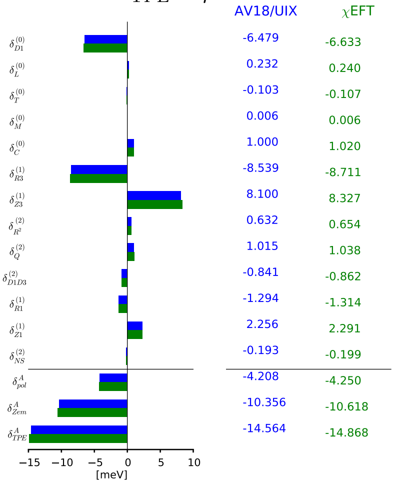
    bound state $\rightarrow$ resonance/continuum
  - Nuclear potentials

    AV18+UIX
    χEFT NN(N^{3}LO)+NNN(N^{2}LO)
    Analyze nuclear-theory uncertainty by comparing $\delta_{\rm pol}$ from different potential models

CJ, Nevo-Dinur, Bacca, Barnea, PRL 111 (2013) 143402
Hernandez, CJ, Bacca, Nevo-Dinur, Barnea, PLB 736 (2014) 344
Nevo Dinur, CJ, Bacca, Barnea, PLB 755 (2016) 380
Hernandez, Ekström, Nevo Dinur, CJ, Bacca, Barnea, PLB 788 (2018) 377
CJ, Bacca, Barnea, Hernandez, Nevo-Dinur, JPG 45 (2018) 093002

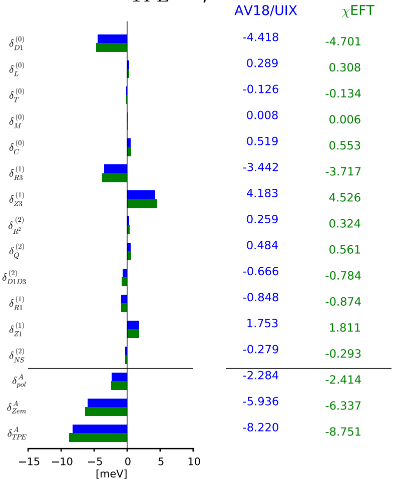
TPE & nuclear polarizability: nuclear-model uncertainty

$\delta_{\rm TPE}$ in $\mu\,^3$He$^+$



$\delta_{\rm TPE} = -14.72$ meV $\pm 1.5\%(1\sigma)$

$\delta_{\rm TPE}$ in $\mu\,^4$He$^+$

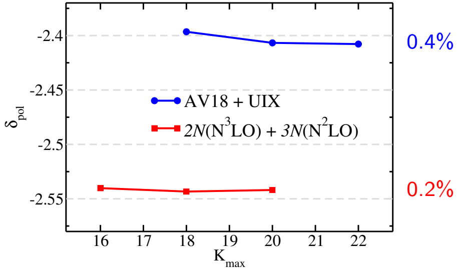


$\delta_{\rm TPE}^{A} = -8.49$ meV $\pm 4.4\%(1\sigma)$

TPE & nuclear polarizability: other uncertainty

Numerical uncertainty

- convergence of EIHH model space (μ^4He$^+$)



Atomic-physics uncertainty

- $(Z\alpha)^6$ correction three-photon exchange
- relativistic and Coulomb distortion effects to sum rules beyond E1
- higher-order nucleonic-structure corrections
- Overall atomic-physics uncertainty
  - 1.5% in μ^3He$^+$
  - 1.3% in μ^4He$^+$

- Combine all uncertainties:

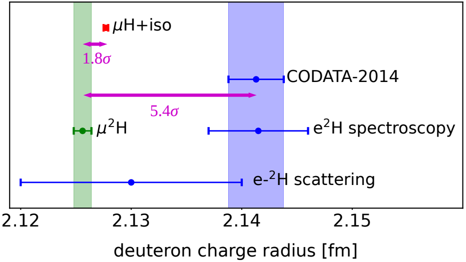
  $\delta_{\rm TPE}(\mu^3{\rm He}^+) = -14.72$ meV $\pm 2.1\%$

  $\delta_{\rm TPE}(\mu^4{\rm He}^+) = -8.49$ meV $\pm 4.6\%$

- The TPE prediction fulfills the 5% accuracy requirements from $\mu^{3,4}$He$^+$ experiments

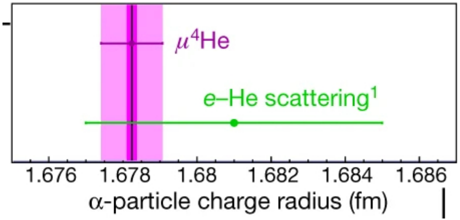
Nuclear charge radii from Lamb shifts in μ^2H and $\mu^{3,4}$He

- Our predictions of nuclear TPE effects have been used by CREMA to extract nuclear charge radii from Lamb shift measurements
- Theoretical uncertainties in TPE effects dominate the error in the extracted nuclear charge radii



$$r_d = 2.12562(13)_{\rm exp}(77)_{\rm theo}\ \text{fm}$$

Pohl, *et al.*, Science (2016)



$$r_\alpha = 1.67824(13)_{\rm exp}(82)_{\rm theo}\ \text{fm}$$

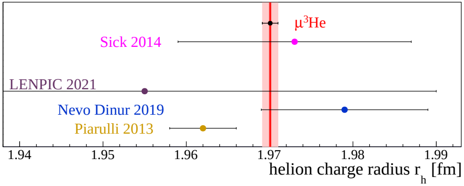
Krauth et al., Nature (2021)

TPE theory:

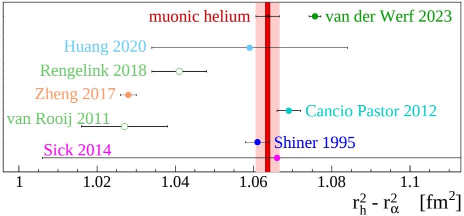
Hernandez, CJ, Bacca, Nevo-Dinur, Barnea, PLB 736 (2014) 344; PRC 100 (2019) 064315 (μ^2H)
Hernandez, Ekström, Nevo Dinur, CJ, Bacca, Barnea, PLB 788 (2018) 377 (μ^2H)
CJ, Nevo-Dinur, Bacca, Barnea, PRL 111 (2013) 143402 (μ^4H)
CJ, Bacca, Barnea, Hernandez, Nevo-Dinur, JPG 45 (2018) 093002 ($\mu^{2,3}$H, $\mu^{3,4}$He$^+$)

Nuclear charge radii from Lamb shifts in μ^2H and $\mu^{3,4}$He

- Our predictions of nuclear TPE effects have been used by CREMA to extract nuclear charge radii from Lamb shift measurements
- Theoretical uncertainties in TPE effects dominate the error in the extracted nuclear charge radii



$$r_h = 1.97007(12)_{\rm exp}(93)_{\rm theo}\ \text{fm}$$

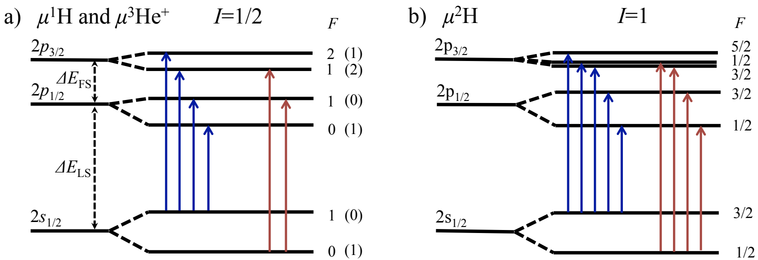


$$r_h^2 - r_\alpha^2 = 1.0636(6)_{\rm exp}(30)_{\rm theo}\ \text{fm}^2$$

Schuhmann et al. (CREMA) arXiv:2305.11679

TPE theory:
Nevo Dinur, CJ, Bacca, Barnea, PLB 755 (2016) 380 (μ^3H, μ^3He$^+$)
CJ, Bacca, Barnea, Hernandez, Nevo-Dinur, JPG 45 (2018) 093002 ($\mu^{2,3}$H, $\mu^{3,4}$He$^+$)

Nuclear Zemach radii from hyperfine splittings in muonic atoms



- Zemach radius R_Z is determined by both nuclear charge and magnetic densities

$$R_Z = \iint d\boldsymbol{r} d\boldsymbol{r}' \rho_E(\boldsymbol{r}) \rho_M(\boldsymbol{r}') |\boldsymbol{r} - \boldsymbol{r}'|$$

- CREMA (PSI): determine R_Z from measured HFS in muonic atoms

Status of theoretical and experimental studies

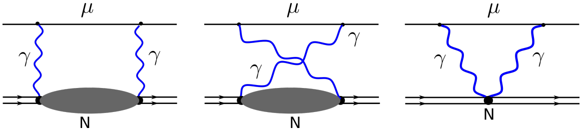
- TPE effects dominate the discrepancy between measured and QED-predicted HFS
- Accidental agreement between the predicted and measured TPE effects in ^{2}H HFS?
- Large discrepancies between the calculated and experimental TPE effects in μ^2H HFS
- Current theories do not rigorously treat nuclear excitations in TPE to HFS.

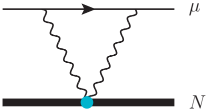
e^2H 1S $E_{HFS}(2\gamma)$ [kHz]	
$\nu_{\mathrm{exp}} - \nu_{\mathrm{qed}}$	45 [1]
Khriplovich, Milstein 2004	43 (model dependent)
Friar 2005	46 $(+18)$
	$(1N$ pol/recoil)

μ^2H 2S $E_{HFS}(2\gamma)$ [meV]	
$\nu_{\mathrm{exp}} - \nu_{\mathrm{qed}}$	0.0966(73) [2]
Kalinowski, Pachucki 2018	0.0383

[1] Wineland, Ramsey, PRA (1972)
[2] Pohl et al., Science (2016)

TPE contributions to HFS in ^{2}H and μ^2H





- TPE effects

$$E_{\rm TPE} = E_{\rm el} + E_{\rm pol} + E_{\rm 1N}$$

- elastic: $F_c(q)$, $F_m(q)$, $F_Q(q)$
- inelastic: vector polarization
- $E_{\rm 1N}$: single-nucleon TPE

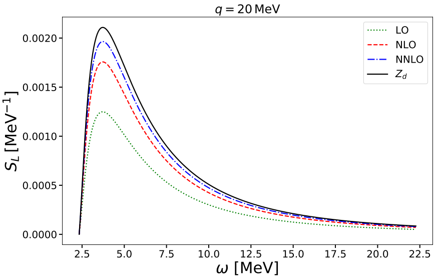
$$\delta^{(0,1)}_{\rm pol} \propto \int d\omega \int dq h^{(0,1)}(\omega,q) S^{(0,1)}(\omega,q)$$

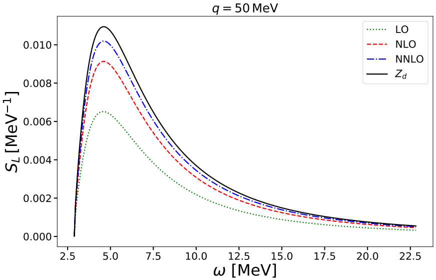
$$S^{(0)}(\omega,q) = -\frac{1}{q^2}{\rm Im} \sum\!\!\!\!\!\!\!\int_{N\neq N_0} \int \frac{d\hat{q}}{4\pi} \langle N_0 II| \left[\vec{q}\times \vec{J}^{\dagger}_m(\vec{q})\right]_3 |N\rangle\langle N|\rho(\vec{q})|N_0 II\rangle \delta(\omega - \frac{q^2}{2m_A} - \omega_N)$$

$$S^{(1)}(\omega,q) = -{\rm Im} \sum\!\!\!\!\!\!\!\int_{N\neq N_0} \int \frac{d\hat{q}}{4\pi} \epsilon^{3jk} \langle N_0 II|\vec{J}^{\dagger}_{m,j}(\vec{q})|N\rangle\langle N|\vec{J}_{c,k}(\vec{q})|N_0 II\rangle |N_0 II\rangle \delta(\omega - \frac{q^2}{2m_A} - \omega_N)$$

- Plan A:χEFT (in progress)
- Plan B:$\not\pi$EFT CJ, Zhang, Platter, arXiv:2311.13585 (today)

$\not\pi$EFT calculation of TPE effects to Lamb shift in ${}^2\mu$H





- longitudinal response function shows order-by-order convergence in $\not\pi$EFT
- TPE predicted in $\not\pi$EFT at NNLO agrees well the χEFT calculations

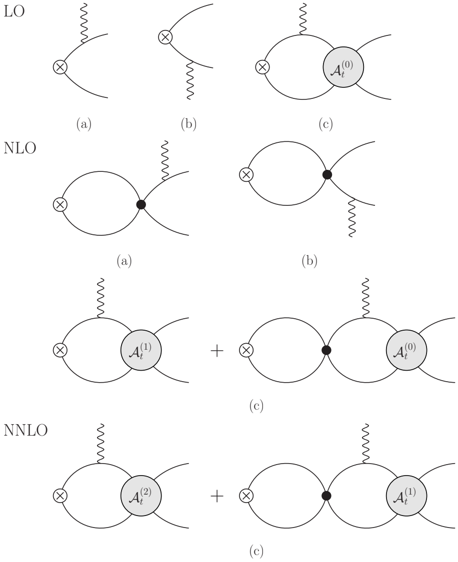
δ_{pol}	non-relativistic kernel	relativistic kernel
$\not\pi$EFT	-1.605	-1.574
χEFT	-1.590	-1.560

Emmons, CJ*, Platter, J. Phys. G 48, 035101 (2021)

$\not\pi$EFT calculation of TPE effects to HFS in ^{2}H and $^2\mu$H

- Contributions from one-body charge density, convection and magnetic curents ρ_E, $\vec{J}_c$, $\vec{J}_m$
-

$$\mathcal{L}_{\mathrm{EM},1b} = -eN^\dagger \frac{1+\tau_3}{2} N A_0 - \frac{ie}{2m_N}\left[N^\dagger \overleftrightarrow{\nabla} \frac{1+\tau_3}{2} N\right] \cdot \vec{A} + \frac{e}{2m_N} N^\dagger (\kappa_0 + \kappa_1 \tau_3)\, \vec{\sigma} \cdot \vec{B} N$$

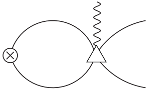


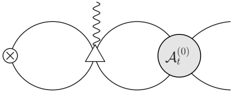
$\not{\pi}$EFT calculation of TPE effects to HFS in ^{2}H and $^2\mu$H

- $\vec{J}_c$ (NLO), $\vec{J}_m$ (NNLO) two-nucleon currents

$$\mathcal{L}_{2,C} = ie\frac{C_2}{4}\left[(N^T P_i N)^\dagger (N^T \overleftrightarrow{\nabla} P_i \tau_3 N) + \text{h.c.}\right] \cdot \vec{A}$$

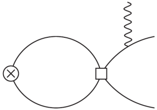
$$\mathcal{L}_{2,B} = -ieL_2\epsilon_{ijk}\left(N^T P_i N\right)^\dagger \left(N^T P_j N\right) B_k + \text{h.c.}$$

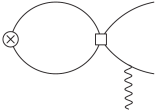


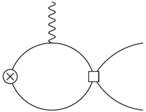


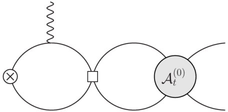
- np S-D mixing at NNLO

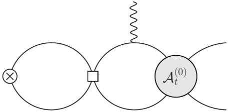
$$\mathcal{L}_{2,Q} = -eL_Q\left(N^T P_i N\right)^\dagger \left(N^T P_j N\right)\left(\nabla^i\nabla^j - \frac{1}{3}\nabla^2\delta_{ij}\right) A_0$$





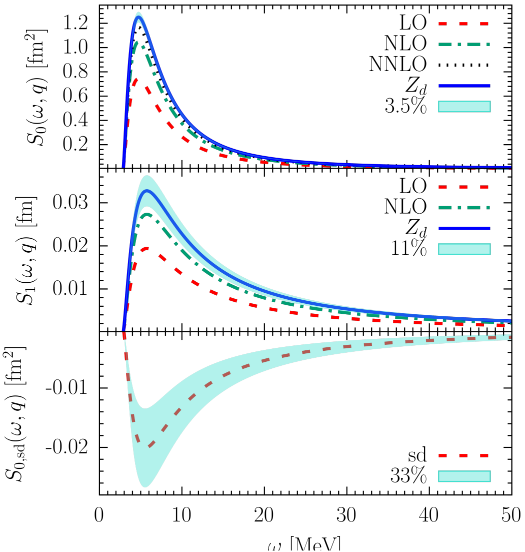






Response functions in $\not\pi$EFT

- $S^{(0)}(\omega, q)$: charge-magnetic transition (LO)
- $S^{(1)}(\omega, q)$: convection-magnetic transition (NLO)
- $S^{(0)}_{\rm sd}(\omega, q)$: S-D mixing correction to $S^{(0)}$ (NNLO)
- systematic order-by-order convergence



TPE corrections to HFS in ^{2}H and μ^2H

	^{2}H (1S)	μ^2H (1S)	μ^2H (2S)
E_{1p} (Antognini 2022)	$-35.54(8)$	$-1.018(2)$	$-0.1272(2)$
E_{1n} (Tomalak 2019)	$9.6(1.0)$	$0.08(3)$	$0.010(4)$
E_{el}	$-41.9(1.5)$	$-0.985(34)$	$-0.123(4)$
E_{pol}	$109.8(3.8)$	$2.86(10)$	$0.358(13)$
E_{TPE}	kHz	meV	meV
This work	$41.7(2.6)$	$0.940(73)$	$0.118(9)$
Khriplovich, Milstein 2004	43		
Friar, Payne 2005 $_{\rm mod}$	64.5		
Kalinowskim, Pauckci 2018		$0.304(68)$	$0.0383(86)$
$\nu_{exp} - \nu_{qed}$	45		$0.0966(73)$

- Consistent with $\nu_{exp} - \nu_{qed}$ within $1.4 - 1.7\sigma$
- Further improvement on accuracy in nuclear theory is demanding
- Uncertainty in E_{1p} and E_{1n} can be larger than expected! (χPT v.s. dispersion)

CJ, Zhang, Platter, arXiv:2311.13585

Conclusion

- radius puzzle & spectroscopy in hydrogen-like atoms
  - Challenge higher-order QED theory
  - TPE effects connect atomic transition with photo-nuclear reaction
  - Use low-energy nuclear theory to probe precision physics
- TPE effects to Lamb shift
  - determine nuclear charge radii
  - Ab inito calculations improve theoretical accuracy to percentage
  - more accurate than extracting information from photonuclear reaction data
- TPE effects to hyperfine splitting
  - determine nuclear magnetic structure
  - Ab initio theory to determine TPE effects to HFS
  - further improve accuracy in nuclear theory (χEFT, or $\not{\pi}\mathrm{EFT}$ at N^{3}LO)
  - uncertainty in nucleonic TPE needs to be reanalyzed
  - Future extension to study TPE effects to HFS in μ^3He, $e^{6,7}$Li

Collaborators

O.J. Hernandez, S. Bacca, T. Richardson	Johannes Gutenberg-Universität Mainz
N. Nevo-Dinur	TRIUMF
N. Barnea	Hebrew University of Jerusalem
A. Ekström	Chalmers University of Technology
S. Pastore, M. Piarulli	Washington University
R.B. Wiringa	Argonne National Laboratory
J.L. Bonilla, L. Platter	University of Tennessee, Knoxville
S.B. Emmons	Carson-Newman University

Pionless effective field theory

- Contact NN and NNN interactions (without pion)
- Predictions require only a few input parameters: a_t, r_t at NNLO (4% accuracy)

$$\begin{aligned}\mathcal{L} \;=\; & N^\dagger\left[i\partial_0+\frac{\nabla^2}{2M}\right]N - C_0\left(N^T P_i N\right)^\dagger\left(N^T P_i N\right)\\ & +\frac{1}{8}C_2\left[\left(N^T P_i N\right)^\dagger\left(N^T \overleftrightarrow{\nabla}^2 P_i N\right)+h.c.\right]-\frac{1}{16}C_4\left(N^T \overleftrightarrow{\nabla}^2 P_i N\right)^\dagger\left(N^T \overleftrightarrow{\nabla}^2 P_i N\right)\\ & +\frac{1}{4}C_0^{(sd)}\left\{\left(N^T P^i N\right)^\dagger\left[N^T P^j\left(\overleftrightarrow{\nabla}_i\overleftrightarrow{\nabla}_j-\frac{1}{3}\delta_{ij}\overleftrightarrow{\nabla}^2\right)N\right]+\text{h.c.}\right\}\end{aligned}$$

Kaplan, Savage, Wise, Nuclear Physics B 534 (1998) 329

- reproduce np $3S1$ phase shift

$$p\cot\delta_t(p) = -\gamma+\frac{\rho}{2}(p^2+\gamma^2)+\cdots$$

$$\begin{aligned}C_0 &= C_{0,-1}+C_{0,0}+C_{0,1}+\cdots\\ C_2 &= C_{2,-2}+C_{2,-1}+\cdots\\ C_4 &= C_{4,-3}+\cdots\end{aligned}$$

$$C_{0,-1} = -\frac{4\pi}{m_N}\frac{1}{\mu-\gamma}, \qquad C_{0,0} = \frac{2\pi}{m_N}\frac{\rho\gamma^2}{(\mu-\gamma)^2},$$

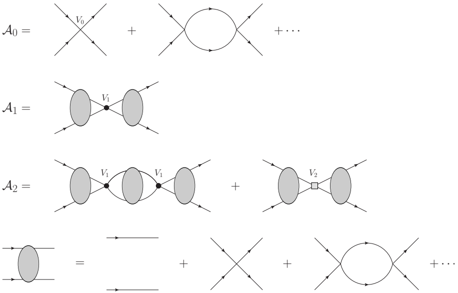
$$C_{0,1} = -\frac{\pi}{m_N}\frac{\rho^2\gamma^4}{(\mu-\gamma)^3}, \qquad C_{2,-2} = \frac{2\pi}{m_N}\frac{\rho}{(\mu-\gamma)^2},$$

$$C_{2,-1} = -\frac{2\pi}{m_N}\frac{\rho^2\gamma^2}{(\mu-\gamma)^3}, \qquad C_{4,-3} = -\frac{\pi}{m_N}\frac{\rho^2}{(\mu-\gamma)^3}$$

$$C_0^{(sd)} = -\frac{6\sqrt{2}\pi}{m_N\gamma^2(\mu-\gamma)}\eta_{sd} \qquad \leftarrow \quad \text{asymptotic D-S ratio}$$

Pionless effective field theory

- Solve Lippmann-Schwinger equation
- t-matrix $\mathcal{A}_n$ in perturbation:



- on-shell:

$$\mathcal{A}_t(p,p;E) = -\frac{4\pi}{m_N}\frac{1}{\gamma + ip}\left[1 + \frac{\rho}{2}(\gamma - ip) + \frac{\rho^2}{4}(\gamma - ip)^2\right]$$

- off-shell:

$$\mathcal{A}_t^{(0)}(k,p;E) = -\frac{4\pi}{m_N}\frac{1}{\gamma + ip}$$

$$\mathcal{A}_t^{(1)}(k,p;E) = -\frac{2\pi}{m_N}\frac{\rho}{\gamma + ip}\left[\gamma - ip + \frac{1}{2(\gamma - \mu)}\left(k^2 - p^2\right)\right]$$

$$\mathcal{A}_t^{(2)}(k,p;E) = -\frac{\pi}{m_N}\frac{\rho^2}{\gamma + ip}\left[(\gamma - ip)^2 + \frac{\gamma - ip}{\gamma - \mu}\left(1 + \frac{\gamma + ip}{\gamma - \mu}\right)\frac{k^2 - p^2}{2}\right]$$